Industries (n) which are composed of plants (i). There is a representative consumer who has nested CES preferences, who is trying to maximize:

$$Y = \sum_n D_n^{\frac{1}{\eta}} Y_n^{\frac{\eta-1}{\eta}} \text{ where } Y_n = \left[\sum_{i\in I_n} D_{ni}^{\frac{1}{\varepsilon_n}} Y_{ni}^{\frac{\varepsilon_n-1}{\varepsilon_n}}\right]^{\frac{\varepsilon_n}{\varepsilon_n-1}},$$

where ε_n is the elasticity of substitution across varieties within and industry and η is the elasticity of substitution across products produced by in different industries. As to what an industry is, think of something like Food, Paper Products, or Transportation Equipment. The D's are preference weights. For this problem (as in Section 2.1 of the paper), I will just solve the "lower nest" part of this problem, looking only at demand of the plant relative to the demand of other plants within that industry;

Given the preferences of the representative consumer, the demand curve of the plant is:

$$Y_{ni} = Y_n \left(P_{ni}/P_n\right)^{-\varepsilon_n} \text{ where } P_n = \left[\sum D_{ni} P_{ni}^{1-\varepsilon_n}\right]^{\frac{1}{1-\varepsilon_n}}$$

is the ideal price index of industry n.

The problem of a plant i is to choose inputs L_{ni}, K_{ni} to maximize profits

$$\begin{aligned} & P_{ni}Y_{ni} - rK_{ni} - wL_{ni}, \text{ where} \\ Y_{ni} &= \left[\left(A_{ni}K_{ni}\right)^{\frac{\sigma_n-1}{\sigma_n}} + \left(B_{ni}L_{ni}\right)^{\frac{\sigma_n-1}{\sigma_n}}\right]^{\frac{\sigma_n}{\sigma_n-1}} \end{aligned}$$

We will add material inputs, which are quantitatively important, later.

For now, define $\alpha_{ni} = \frac{rK_{ni}}{rK_{ni}+wL_{ni}}$ as the capital share of plant i, and θ_{ni} as the $\frac{rK_{ni}+wL_{ni}}{rK_n+wL_n}$ as the expenditure share of plant i, within industry n. Then $\alpha_n = \sum_i \alpha_{ni}\theta_{ni}$ is the capital share of industry n:

What we are interested in are the following two parameters:

$$\begin{aligned} \sigma^{agg} &\equiv 1 + \frac{\partial \log\left(Kr/Lw\right)}{\partial \log\left(w/r\right)}, \text{ and} \\ \sigma_n^N &\equiv 1 + \frac{\partial \log\left(K_n r/L_n w\right)}{\partial \log\left(w/r\right)} \\ &= 1 + \frac{\partial \log\left(\frac{\alpha_n}{1-\alpha_n}\right)}{\partial \log\left(w/r\right)} \\ &= 1 + \frac{1}{\alpha_n\left(1-\alpha_n\right)} \frac{\partial \alpha_n}{\partial \log\left(w/r\right)} \end{aligned}$$

Here, we will just work through the calculations to solve for σ_n^N. The calculations for σ^{agg} are very similar.

Now use the definition of α_n and then the product rule:

$$\begin{aligned}
\sigma_n^N &= 1+\frac{1}{\alpha_n\left(1-\alpha_n\right)}\frac{\partial\left(\sum_i \alpha_{ni}\theta_{ni}\right)}{\partial\log\left(w/r\right)} \\
&= 1+\frac{1}{\alpha_n\left(1-\alpha_n\right)}\left[\sum_i \theta_{ni}\frac{\partial\alpha_{ni}}{\partial\log\left(w/r\right)}+\sum_i \alpha_{ni}\frac{\partial\theta_{ni}}{\partial\log\left(w/r\right)}\right] \\
&= 1+\frac{1}{\alpha_n\left(1-\alpha_n\right)}\left[\sum_i \theta_{ni}\alpha_{ni}\left(1-\alpha_{ni}\right)\left(1-\sigma\right)+\sum_i \alpha_{ni}\theta_{ni}\frac{\partial\log\theta_{ni}}{\partial\log\left(w/r\right)}\right]
\end{aligned}$$

Now what is $\frac{\partial\log\theta_{ni}}{\partial\log(w/r)}$? It is

$$\begin{aligned}
\frac{\partial\log\theta_{ni}}{\partial\log\left(w/r\right)} &= \frac{\partial\log\theta_{ni}}{\partial\log\left(\frac{p_{ni}}{p_n}\right)}\frac{\partial\log\left(\frac{p_{ni}}{p_n}\right)}{\partial\log\left(w/r\right)} \\
&= \left(\varepsilon_n-1\right)\cdot\left(\alpha_{ni}-\alpha_n\right)
\end{aligned}$$

The first term comes from the preferences of the representative consumer. The second comes from the cost minimization condition of each plant. Suppose that the rental rate goes up by one percentage point. Then the increase in the cost of producing for each plant will be proportional to the capital cost share of the plant.

Given this we can substitute $\frac{\partial\log\theta_{ni}}{\partial\log(w/r)}$ to write out the expression for σ_n^N.

$$\begin{aligned}
\sigma_n^N &= 1+\sum_i \theta_{ni}\frac{\alpha_{ni}\left(1-\alpha_{ni}\right)}{\alpha_n\left(1-\alpha_n\right)}\left(\sigma-1\right)+\sum_i \theta_{ni}\frac{\alpha_{ni}\left(\alpha_{ni}-\alpha_n\right)}{\alpha_n\left(1-\alpha_n\right)}\left(\varepsilon_n-1\right) \\
&= 1+\left[1-\sum_i \theta_{ni}\frac{\alpha_n-\alpha_{ni}-\alpha_n^2+\alpha_{ni}^2}{\alpha_n\left(1-\alpha_n\right)}\right]\left(\sigma-1\right)+\sum_i \theta_{ni}\frac{\alpha_{ni}^2-\alpha_{ni}\alpha_n}{\alpha_n\left(1-\alpha_n\right)}\left(\varepsilon_n-1\right) \\
&= 1+\left[1-\underbrace{\sum_i \theta_{ni}\frac{\left(\alpha_{ni}-\alpha_n\right)^2}{\alpha_n\left(1-\alpha_n\right)}}_{\chi}\right]\left(\sigma-1\right)+\underbrace{\sum_i \theta_{ni}\frac{\left(\alpha_{ni}-\alpha_n\right)^2}{\alpha_n\left(1-\alpha_n\right)}}_{\chi}\left(\varepsilon_n-1\right) \\
&= 1+\left(1-\chi\right)\left(\sigma-1\right)+\chi\left(\varepsilon_n-1\right) \\
&= \left(1-\chi\right)\sigma+\chi\varepsilon_n
\end{aligned}$$

In words, the industry-level elasticity of substitution is a convex combination of the plant-level elasticity of substitution and the plant-level elasticity of demand. The demand elasticity parameterizes how sensitive the scale of the plant is to changes in its marginal cost of production. Consider, for example, an increase in the price of the the capital input. The marginal cost of production will increase more for plants with relatively large capital shares, as we talked about. As a result, labor-intensive plants will produce relatively more of the total industry output following the increase of the capital rental rate. The ability to substitute across plants is parameterized by χ which is increasing in the heterogeneity of capital cost shares.