Problem Set 5: Due Wednesday, December 3.

Problem 1

In this problem, we will work through the problem of a firm that invests, but faces different prices in buying versus selling the investment good. At the start of the period t our firm is going to have capital k and productivity A. Its choice is over k', how much capital to have at the beginning of the following period. Assume that the value function of the firm is given by:

$$\begin{aligned} \hat{V}(A,k) &= \max_{k'} \frac{r}{\alpha(1+r)} A^{1-\alpha}(k')^{\alpha} - (k'-k) + \mathbf{1}_{k'<k}(1-p_s)(k'-k) \\ &\quad + \frac{1}{1+r}\mathbb{E}\left[\hat{V}(A',k')\right] \end{aligned} \tag{1}$$

Note that, according to this formulation, capital is installed instantaneously (there is no one-period lag), and there is no depreciation. Also, $p_s < 1$ is the price at which the firm can sell the capital good, and 1 is the price at which the firm can buy the capital good. Finally, $\frac{1}{1+r}$ is the discount factor.

1. Let $z \equiv \frac{k}{A}, z' \equiv \frac{k'}{A}, \gamma \equiv \frac{A'}{A}$, and $V(A,z) \equiv \hat{V}(A,Az)$. Show that, according to these definitions, Equation (1) is equivalent to

$$V(A,z) = \max_{z'} \frac{r}{\alpha(1+r)} A(z')^{\alpha} + (z-z')A + \mathbf{1}_{k'<k}(1-p_s)A(z'-z) + \frac{1}{1+r}\mathbb{E}\left[V\left(\gamma A, \frac{z'}{\gamma}\right)\right] \tag{2}$$

2. Argue why V is homogeneous of degree 1 in A. Using this fact, and the definition $v(z) \equiv \frac{V(A,z)}{A}$, write out the investment problem of the firm in terms of a single-state value function.[1]

3. Assume $\log\gamma$ (the productivity growth rate) is Δ with probability π, 0 with probability $(1-2\pi)$, and $-\Delta$ with probability π. Assume also that $\pi e^{-\Delta} + (1-2\pi) + \pi e^{\Delta} < 1+r$. Use the assumption on the values that γ can possibly take to write the value function $v(z)$ in terms of $v(z'e^{\Delta})$, $v(z')$, and $v(z'e^{-\Delta})$.

[1] Hint: Your answer, at this point, should be

$$v(z) = \max_{z'} \frac{r}{\alpha(1+r)}(z')^{\alpha} - (z'-z) + \mathbf{1}_{k'<k}(1-p_s)(z'-z) + \frac{1}{1+r}\mathbb{E}\left[\gamma v\left(\frac{z'}{\gamma}\right)\right]$$

4. Conjecture that there is a region of inaction that has endpoints $\underline{z}$ and $\bar{z}$, and that outside of the region of inaction the optimal policy is to go to the closest endpoint. In other words, conjecture that there exists two numbers $\underline{z} < \bar{z}$ such that if $z \in [\underline{z}, \bar{z}]$ then $z' = z$; if $z > \bar{z}$ then $z' = \bar{z}$; and that if $z < \underline{z}$ then $z' = \underline{z}$. Use this conjecture, and your answer to part 3, to re-write your value function $v(z)$, for now just for the z within the inaction region (where $z \in [\underline{z}, \bar{z}]$).

5. Verify that the solution to your answer to part 4 takes the form $v(z; c_1, c_2) = \kappa z^\alpha + c_1 z^{\rho_1} + c_2 z^{\rho_2}$ where $\kappa = r\left[\alpha r + 2\alpha\pi - e^{(1-\alpha)\Delta}\alpha\pi - \alpha\pi e^{-(1-\alpha)\Delta}\right]^{-1}$; and ρ_1 and ρ_2 are the two roots of the equation $\pi e^{-(1-\rho)\Delta} + (r - 2\pi) + \pi e^{(1-\rho)\Delta} = 0$,[2] and where c_1 and c_2 are two constants which we will solve for.

6. Maintain our conjecture that $z' = \bar{z}$ if $z > \bar{z}$. Explain why $\frac{\partial v(z;c_1,c_2)}{\partial z} = p_s \; \forall \; z \geq \bar{z}$. Hint: It should be easy to argue, just with words, why $\frac{\partial v(\bar{z};c_1,c_2)}{\partial z} = p_s$. To get the derivative of the value function for $z > \bar{z}$, start by writing $v(\bar{z})$ and $v(z)$ for an arbitrary $z > \bar{z}$, then take the difference between these two value functions.

7. Using the same logic, explain why $\frac{\partial v(z;c_1,c_2)}{\partial z} = 1 \; \forall \; z \leq \underline{z}$. The equations that you've written in parts 6 and 7 are called the *smooth pasting conditions.*

8. So far we have just taken $\underline{z}$ and $\bar{z}$ as given, not explaining how they are determined. Now suppose that $\underline{z}$ and $\bar{z}$ are chosen optimally (instead of being dictated by some arbitrary rule). From parts 6 and 7, we know that

$$\frac{\partial v(\bar{z}; c_1, c_2)}{\partial z} = p_s; \quad \frac{\partial v(\underline{z}; c_1, c_2)}{\partial z} = 1$$

Totally differentiate the previous two equations:

$$\begin{aligned} \frac{\partial^2 v(\bar{z}; c_1, c_2)}{\partial z^2} + \frac{\partial^2 v(\bar{z}; c_1, c_2)}{\partial z \partial c_1}\frac{\partial c_1}{\partial \bar{z}} + \frac{\partial^2 v(\bar{z}; c_1, c_2)}{\partial z \partial c_2}\frac{\partial c_2}{\partial \bar{z}} &= 0 \\ \frac{\partial^2 v(\underline{z}; c_1, c_2)}{\partial z^2} + \frac{\partial^2 v(\underline{z}; c_1, c_2)}{\partial z \partial c_1}\frac{\partial c_1}{\partial \underline{z}} + \frac{\partial^2 v(\underline{z}; c_1, c_2)}{\partial z \partial c_2}\frac{\partial c_2}{\partial \underline{z}} &= 0 \end{aligned}$$

[2] The solutions to the latter equation are:

$$\begin{aligned} \rho_1 &= \frac{1}{\Delta}\log\left[\frac{e^\Delta}{2}\left[\frac{r}{\pi} - \sqrt{\left(\frac{r}{\pi}\right)^2 + \frac{4r}{\pi}}\right]\right] \\ \rho_2 &= \frac{1}{\Delta}\log\left[\frac{e^\Delta}{2}\left[\frac{r}{\pi} + \sqrt{\left(\frac{r}{\pi}\right)^2 + \frac{4r}{\pi}}\right]\right] \end{aligned}$$

A necessary condition of choosing the cutoffs optimally is that a small change in $\underline{z}$ and $\bar{z}$ won't change the value of c_1 or c_2. In other words, if $\underline{z}$ and $\bar{z}$ are chosen optimally, then $\frac{\partial c_1}{\partial \bar{z}} = \frac{\partial c_2}{\partial \bar{z}} = \frac{\partial c_1}{\partial \underline{z}} = \frac{\partial c_2}{\partial \underline{z}} = 0$.

Thus, at the optimum $\underline{z}$ and $\bar{z}$,

$$\frac{\partial^2 v(\bar{z}; c_1, c_2)}{\partial z^2} = 0 \text{ and } \frac{\partial^2 v(\underline{z}; c_1, c_2)}{\partial z^2} = 0 \qquad (3)$$

These are called the *super contact conditions*. Use these two equations, in addition to the two equations you wrote out in parts 6 and 7 (namely $\frac{\partial v(\bar{z}; c_1, c_2)}{\partial z} = p_s$ and $\frac{\partial v(\underline{z}; c_1, c_2)}{\partial z^2} = 1$) to solve for the four unknowns c_1, c_2, $\underline{z}$, and $\bar{z}$.[3]

Once you have solved for c_1, c_2, $\underline{z}$, and $\bar{z}$, you can use parts 5-7 to write out $v(z)$ only in terms of parameters. Do so using $(p_s, r, \alpha, \pi, \Delta) = (0.5, 0.1, 0.3, 0.03, 0.01)$. Plot out $v(z)$ as a function of z.

Problem 2

I have sent you, via e-mail, links to four job market papers, written by your peers from other schools. Write a two-to-three page referee report on the paper of your choice. As a reminder, here is a link to Elisabeth Sadoulet and Alain de Janvry's guidelines on how to write a referee report:

http://are.berkeley.edu/courses/ARE251/2004/assignments/RRGuidelines.pdf .

[3] Hint: There are many ways to numerically solve for these four unknowns. One relatively easy way that I found was using *Mathematica*. You can use the Solve command to solve for c_1 and c_2 for arbitrary values of $\underline{z}$ and $\bar{z}$ using the equations from parts 6 and 7. Then with c_1 and c_2 solved for, you can solve for $\underline{z}$ and $\bar{z}$ using Equation (3) and the FindRoot command.