Notes on Gabaix (2011): "The Granular Origins of Aggregate Fluctuations"

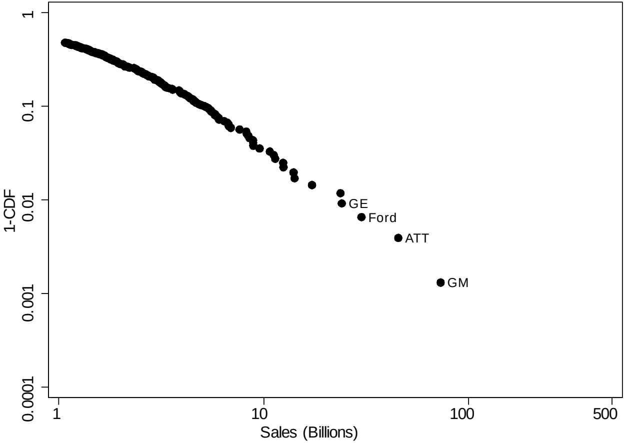
Research question and motivation

- Majority of dynamic general equilibrium models: Firm (scale) heterogeneity does not matter.
- Because some firms are so large, decisions of individual firms can have aggregate implications
  - 2004Q4: Microsoft issues $24 billion one-time dividend. Accounts for 2.1% boost in personal income growth.
  - 2000: Nokia accounts for *half* of Finish private R&D, 1.6 percentage points of GDP growth.
  - Are these anecdotes exceptional or common?
- Question: To what extent are firm-level shocks responsible for aggregate fluctuations?

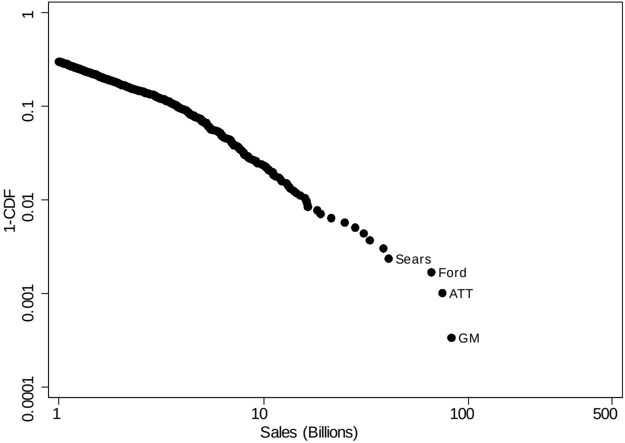
Outline

- Some data
  - Compustat: 1960 to present.
- Theoretical results and calibration
  - The Central Limit Theorem is irrelevant when firm sizes are fat-tailed
  - The herfindahl index is a summary statistic for the importance of firm-specific shocks.
- The granular residual

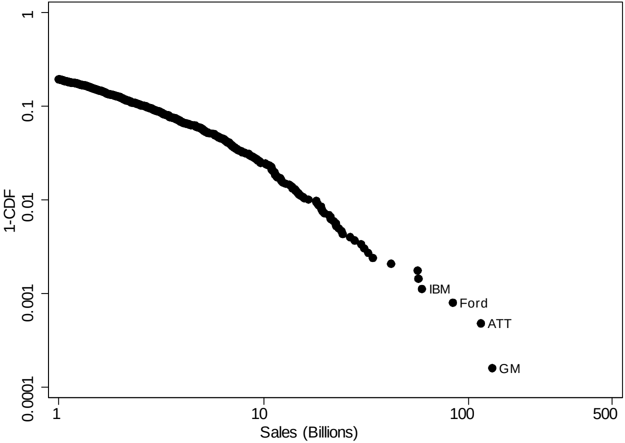
The firm size distribution in 1960



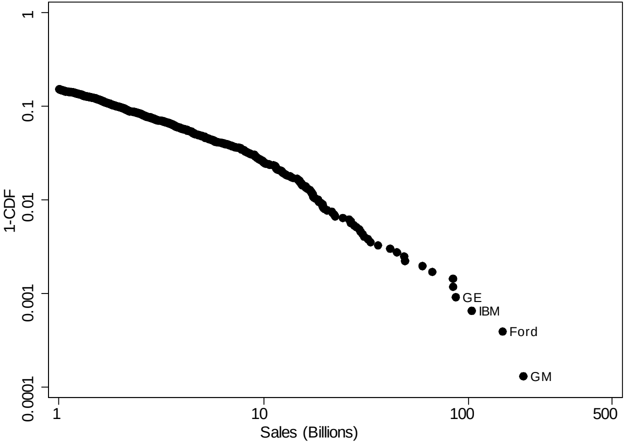
The firm size distribution in 1970



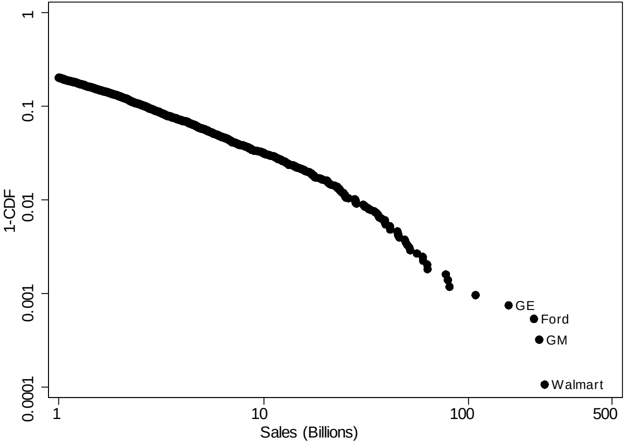
The firm size distribution in 1980



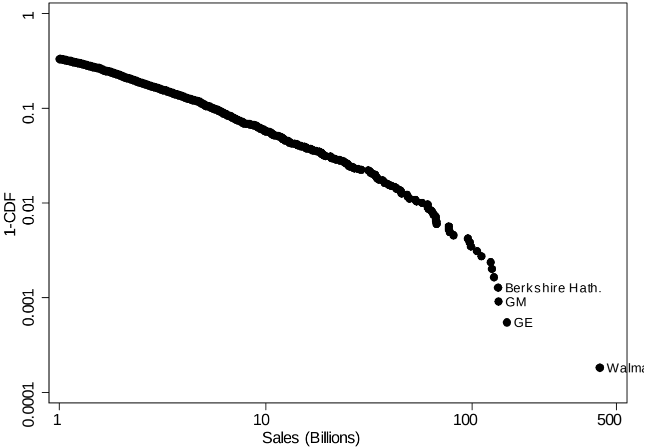
The firm size distribution in 1990



The firm size distribution in 2000

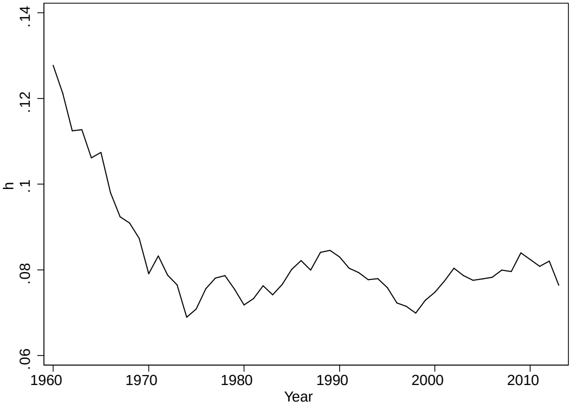


The firm size distribution in 2010



Sales Herfindahl of firms in Compustat

$$h = \left[\sum_i \left(\frac{S_i}{S} \right)^2 \right]^{1/2}$$



Overview of the theoretical results

- If the firm size distribution is Pareto, we can show how the dispersion of GDP growth decreases in economies with more and more firms.
- Even if the firm size distribution is not Pareto, we can relate the dispersion of GDP to:
  - σ: the standard deviation of firm productivity growth rates.
  - h: the HHI of firm sales
  - a combination of other model parameters.

"Islands Economy"

What is the relationship between micro productivity growth and aggregate output growth?

- Economy is made up of n units (firms or industries)
- Utility$=\log C$, where $C \equiv \prod_{i=1}^{n} \left(\frac{C_i}{\xi_i}\right)^{\xi_i}$
- $L \equiv \sum_i L_i$ is fixed at 1
- Set C as the numeraire good: $P \equiv \prod_i (P_i)^{\xi_i} = 1$
- Production : $C_i = A_i L_i$
  - σ =SD of productivity shocks, which are i.i.d. across units.
- One can show:
  - $\frac{P_i C_i}{C} = \xi_i$
  - $\log C = \sum_i \frac{P_i C_i}{C} \log(A_i)$
- Thus:
  - $\text{Var}(\log C) = \sum \frac{P_i C_i}{C} \sigma^2$
  - $\sigma_{\log C} = \text{SD}(\log C) = \underbrace{\left[\sum \left(\frac{P_i C_i}{C}\right)^2\right]^{1/2}}_{\equiv h} \cdot \sigma$

The Pareto Distribution

Let $S_i \equiv P_i C_i$ be a Pareto(ζ,x_0) random variable.
$\mathrm{P}(S > x) = \left(\frac{x}{x_0}\right)^{-\zeta}$.

Some useful facts about the Pareto distribution:

- $\mathbb{E}\left[S\right] = x_0 \frac{\zeta}{\zeta - 1}$ if $\zeta > 1$, ∞ otherwise
- $\mathbb{E}\left[S^2\right] = (x_0)^2 \frac{\zeta}{\zeta - 2}$ if $\zeta > 2$, ∞ otherwise
- S^α is Pareto $\left(\frac{\zeta}{\alpha},\ (x_0)^\alpha\right)$ distributed.
- αS is Pareto $(\zeta,\ \alpha x_0)$ distributed.
- r^{th} moment of the k^{th} largest value in a sample of $N \equiv \mathbb{E}\left[S^r_{k:N}\right] = (x_0)^r \frac{\Gamma\left[k - \frac{r}{\zeta}\right]}{\Gamma[k]} \frac{\Gamma[N+1]}{\Gamma\left(N+1-\frac{r}{\zeta}\right)}$, if $r > \zeta$.
- Many other facts in Gabaix (2009, Section 2)

Classic Central Limit Theorem

Suppose S_1, S_2, ... S_N is a sequence of i.i.d. random variables with $\mathbb{E}[S_i] = \mu$ and $\text{Var}[S_i] = \sigma^2 < \infty$. Then, as N approaches ∞,

$$\frac{\sqrt{N}}{\sigma}\left(\frac{\sum S_i}{N} - \mu\right) \to_{\text{d}} \mathcal{N}(0,1)$$

"Compared to an economy with 10 firms, an economy with 10 million firms will have GDP growth with a standard deviation 0.1% as large".

What if $\text{Var}[S_i] = \infty$?

Central Limit Theorem with infinite variances

Suppose S_1, S_2, ... S_N is a sequence of i.i.d. nonnegative random variables with $\mathrm{P}(S_i > x) = x^{-\zeta} L(x)$ (where $L(x)$ is a *slowly-varying function*, and $\zeta < 2$). Then

$$\left(\frac{\sum_i S_i - b_N}{a_N} \right) \to \mathcal{L}(\zeta), \text{ where}$$

$$a_n = \inf \left\{ x : P(S_i > x) \leq \frac{1}{N} \right\} \text{ and } b_n = N \mathbb{E}\left[S_i \cdot 1_{(X_i \leq a_n)} \right]$$

and $\mathcal{L}(\zeta)$ is a *Levy distribution* with exponent ζ.

- PDF of Levy distribution: $\sqrt{\frac{\zeta}{2\pi}} \exp\left\{ -\frac{\zeta}{2x} \right\} x^{-3/2}$
- A slowly-varying function, $L(x)$ is one that satisfies $\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1 \ \forall \ t > 0$.
- If $\mathrm{P}(S_i > x) = \left(\frac{x}{x_0} \right)^{-\zeta}$, then

  $$a_n = \inf \left\{ x : \left(\frac{x}{x_0} \right)^{-\zeta} \leq \frac{1}{N} \right\} = x_0 N^{1/\zeta}, \ b_N = 0$$
- Thus $\frac{N^{1-1/\zeta}}{x_0} \frac{\sum S_i}{N} \to \mathcal{L}(\zeta)$

Proposition 2

"Consider a series of island economies indexed by N. Economy N has N firms whose growth rate volatility is σ and whose sizes $S_1,..S_N$ are independently drawn from a power law distribution."

$$P(S > x) = ax^{-\zeta}, \text{ with } \zeta \geq 1.$$

As $N \to \infty$, GDP volatility follows

$$\sigma_{GDP} \sim \frac{v_\zeta}{\log N}\sigma \text{ for } \zeta = 1$$
$$\sigma_{GDP} \sim \frac{v_\zeta}{N^{1-1/\zeta}}\sigma \text{ for } \zeta \in (1,2)$$
$$\sigma_{GDP} \sim \frac{v_\zeta}{N^{1/2}}\sigma \text{ for } \zeta \geq 2$$

When $\zeta \geq 2$, v_ζ is a constant; when $\zeta < 2$, v_ζ is the square root of a Levy distributed (with exponent $\zeta/2$) random variable.

Intuition for Proposition 2

In our islands economy, $\sigma_{GDP} = \sigma h$. Looking across economies with different numbers of firms, how does h change as N changes?

Take $P(S > x) = ax^{-\zeta}$, and consider the case in which $\zeta \in (1, 2)$, and $a = 1$.

$$\frac{\mathbb{E}[X_{k:N}]}{N\mathbb{E}[X]} = \frac{\Gamma\left[k - \frac{1}{\zeta}\right](\zeta - 1)}{\Gamma[k]\zeta} \frac{\Gamma[N]}{\Gamma\left(N + 1 - \frac{1}{\zeta}\right)}$$

$$\to_{N\to\infty} \frac{\Gamma\left[k - \frac{1}{\zeta}\right](\zeta - 1)}{\Gamma[k]\zeta} N^{-(1-1/\zeta)}$$

Share of top K firms is proportional to $N^{-(1-1/\zeta)} \Rightarrow h$ is proportional to $N^{-(1-1/\zeta)}$.

Proof of Proposition 2, Part 1

If $\zeta \geq 2$, the variance of S_i is finite. Can apply the formula $\sigma_{GDP} = \sigma h$

$$h = \frac{1}{N^{1/2}} \frac{\left[N^{-1} \sum (S_i)^2\right]^{1/2}}{N^{-1} \sum S_i}$$

$$\sigma_{GDP} \rightarrow \frac{\sigma}{N^{1/2}} \cdot \frac{\left(\mathbb{E}\left[S^2\right]\right)^{1/2}}{\mathbb{E}\left[S\right]}$$

Proof of Proposition 2, Part 2

When $\zeta > 1$, $N^{-1} \sum S_i \to \mathbb{E}[S]$
S_i^2 has a power law exponent $\zeta/2$

$$P\left((S_i)^2 > x\right) = ax^{-\zeta/2}$$

Use the CLT with infinite variances, if $\zeta > 1$

$$N^{-2/\zeta} \sum S_i^2 \to_d \mathcal{L}(\zeta/2)$$

$$N^{1-1/\zeta} h = N^{1-1/\zeta} \frac{\left[N^{-2/\zeta} \left(\sum S_i^2\right)\right]^{1/2}}{N^{-1} \sum S_i} \to_d \frac{(\mathcal{L}(\zeta/2))^{1/2}}{\mathbb{E}[S]}$$

Putting the pieces together

$$\sigma_{GDP} N^{1-1/\zeta} = \sigma h N^{1-1/\zeta} \to_d \sigma \frac{(\mathcal{L}(\zeta/2))^{1/2}}{\mathbb{E}[S]}$$

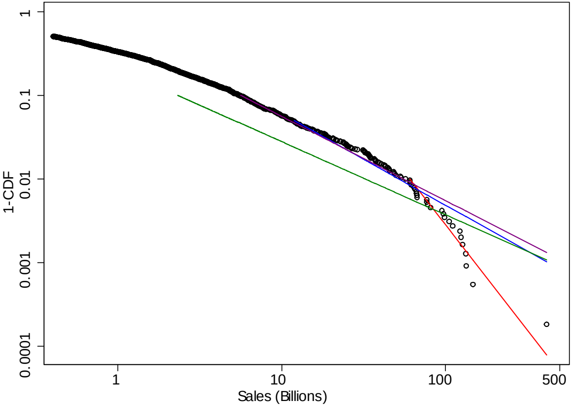
If $\zeta \approx 1.05 \Rightarrow N^{1-1/\zeta} \approx N^{0.05} \Rightarrow$ Compared to an economy with 10 firms, an economy with 10 million firms will have GDP growth with a standard deviation about half as large.

Digression: Is the firm size distribution Pareto?

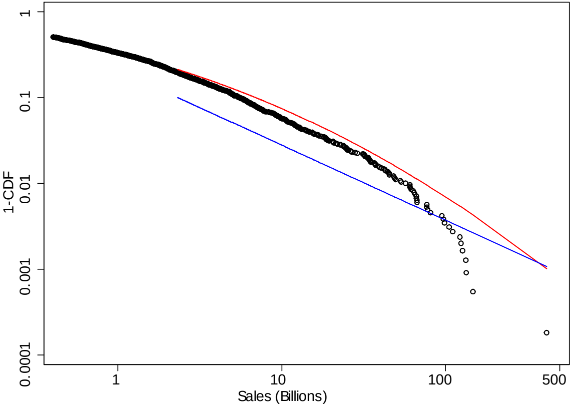
- With moderate sample size it's difficult to distinguish between Pareto distribution (which has finite variance) and something like a lognormal distribution (for which regular CLT applies).
- Find best fit, assuming firm sizes are distributed either Pareto or lognormal.
- $f(x) = \frac{\zeta (x_0)^\zeta}{x^{\zeta+1}} \Rightarrow \log f(x) = \log \zeta + \zeta \log x_0 - (\zeta + 1) \log x$
- $\frac{\partial \log \mathcal{L}}{\partial \hat{\zeta}} = \sum_{i=1}^{n} \frac{1}{\zeta} + \log\left(\frac{x_0}{x}\right) = 0 \Rightarrow \hat{\zeta} = \left[\frac{1}{N} \sum \log\left(\frac{x}{x_0}\right)\right]^{-1}$

Sample	$\hat{x}_0$	$\hat{\zeta}$
80	2.32	0.87
90	5.62	1.00
95	11.96	1.10
99	60.75	2.52

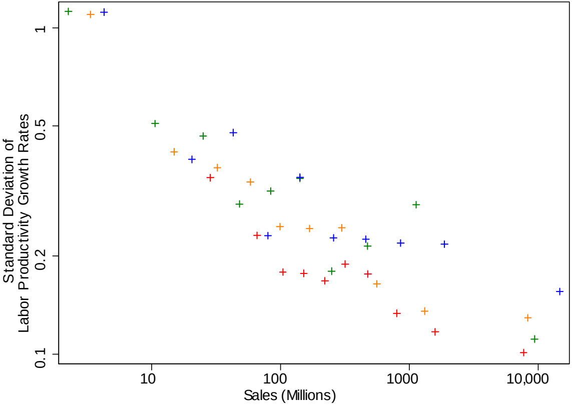
Digression: Is the firm size distribution Pareto?



Digression: Is the firm size distribution Pareto?



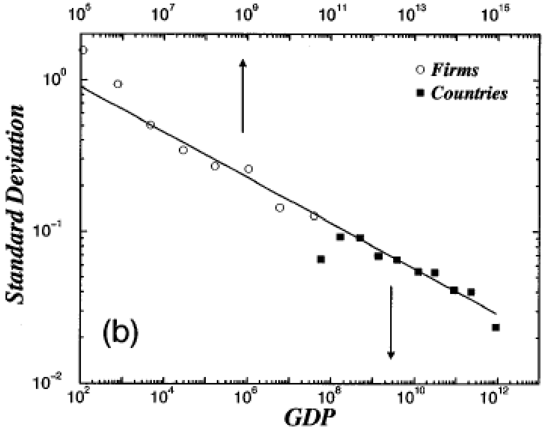
The dispersion of growth rates decreases with size



- $\log\left(\sigma^{\text{Grow}}\right) = \kappa_0 - \kappa_1 \log(\text{size})$;
- $\kappa_1 \in [0.15, 0.25]$, compare to benchmark of perfect correlations of shocks within firms $(\kappa_1 = 0)$ or no correlation $(\kappa_1 = \frac{1}{2})$

The dispersion of growth rates decreases with size

Lee et al. (1998)



We can extend Proposition 2 to allow for firm size and firm volatility to be related.

Consider a series of island economies indexed by N. Economy N has N firms whose growth rate volatility is $\sigma^{\text{firm}}(S) = \sigma\left(\frac{S}{x_0}\right)^{-\alpha}$ and whose sizes $S_1,..S_N$ are independently drawn from a power law distribution.

$$P(S > x) = x^{-\zeta}, \text{ with } \zeta \geq 1.$$

If $\zeta > 1$, the volatility of GDP, $\sigma(Y)$, is proportional to $N^{-\min\left\{\frac{1}{2}, 1-\frac{1-\alpha}{\zeta}\right\}}$.

If $\zeta \approx 1.05$ and $\alpha \approx \frac{1}{6} \Rightarrow N^{1-\frac{1-\alpha}{\zeta}} \approx N^{0.21} \Rightarrow$ Compared to an economy with 10 firms, an economy with 10 million firms will have GDP growth with a standard deviation about 5% as large.

Hulten (1978)

What is the relationship between micro productivity growth and aggregate output growth?

- Economy is made up of n units (firms or industries)
- Utility$= \log C - \frac{\phi}{\phi+1} L^{\frac{\phi+1}{\phi}}$, where $C \equiv \prod_i \left(\frac{C_i}{\xi_i}\right)^{\xi_i}$
- Production : $Q_i = A_i \left(\left(\frac{L_i}{\alpha b}\right)^{\alpha} \left(\frac{K_i}{(1-\alpha)b}\right)^{1-\alpha} \right)^{b} \left(\frac{M_i}{1-b}\right)^{1-b}$
- Intermediate input bundle: $M_i = \prod_j (M_{j\to i})^{\gamma_{ji}}$
- Market clearing: $Q_i = C_i + \sum_j X_{i\to j} + M_{i\to j}$
  - Version 1: Capital is fixed across time.
  - Version 2: $K_i = \prod_j (X_{j\to i})^{\theta_{ji}}$
- Write
  - P_i as the Lagrange multiplier for the good i market-clearing condition, and $S_i \equiv P_i Q_i$.
  - W as the Lagrange multiplier for the labor market clearing condition.
  - Set C as the numeraire good: $P \equiv \prod_i (P_i)^{\xi_i} = 1$

Hulten (1978)

Step 1: Solve for Total Labor Supply

Consider the problem of the representative consumer who is trying to maximize:

$$\log C - \frac{\phi}{\phi + 1} L^{\frac{\phi+1}{\phi}} \text{ s.t. } C = WL$$

Equilibrium C and L satisfy:

$$L = W^{\phi}$$

$$C = W^{\phi+1} = L^{\frac{\phi+1}{\phi}} \tag{1}$$

Hulten (1978)

Step 2: Solve for Prices

Consider the cost-minimization problem of firm/industry i.

Version 1:

$$\log P_i = -\log A_i + (1-\alpha)\, b \log \left(K_i / \left[(1-\alpha)\, b\right]\right) + \alpha b \log W$$
$$+ (1-\alpha)\, b \log S_i + (1-b) \sum_j \gamma_{ji} \log P_j$$

$$\overrightarrow{\log P} = \left(I - (1-b)\,\Gamma'\right)^{-1} \{-\overrightarrow{\log A} - (1-\alpha)\, b\overrightarrow{\log \left(K / \left[(1-\alpha)\, b\right]\right)}$$
$$+ \alpha b \log W + (1-\alpha)\, b\overrightarrow{\log S}\}$$

Version 2:

$$\log P_i = -\log A_i + \alpha b \log W + \sum_j \left[(1-\alpha)\, b\theta_{ji} + (1-b)\,\gamma_{ji}\right] \log P_j$$

$$\overrightarrow{\log P} = \left(I - (1-b)\,\Gamma' - (1-\alpha)\, b\Theta'\right)^{-1} \left(-\overrightarrow{\log A} + \alpha b \log W\right) \tag{2}$$

Hulten (1978)

Step 3: Write out sales in each industry

Using the market clearing conditions (Version 1)

$$S_i = P_i Q_i = P_i C_i + \sum_j P_i M_{i \to j}$$

Plugging in customers' factor demand curves and re-arranging:

$$S_i - (1-b) \sum_j \gamma_{ij} S_j = \xi_i C$$

$$\overrightarrow{S} = (I - (1-b)\,\Gamma)^{-1}\, \overrightarrow{\xi}\, C$$

In Version 2:

$$S_i = P_i C_i + \sum_j P_i M_{i \to j} + P_i X_{i \to j}$$

which eventually yields

$$\overrightarrow{S} = (I - (1-b)\,\Gamma - (1-\alpha)\, b\Theta)^{-1}\, \overrightarrow{\xi}\, C \tag{3}$$

Hulten (1978)

Step 4: Write out total consumption and labor in terms of productivity

In Version 2: Plug Equation (2) into Equation (1)

$$\overrightarrow{\log P} = \left(I - (1-b)\,\Gamma' - (1-\alpha)\,b\Theta'\right)^{-1}\left(-\overrightarrow{\log A} + \alpha b \log W\right)$$
$$= \left(I - (1-b)\,\Gamma' - (1-\alpha)\,b\Theta'\right)^{-1}\left(-\overrightarrow{\log A} + \frac{\alpha b}{\phi+1}\log C\right)$$

Use the fact that $\xi'\overrightarrow{\log P} = 0$

$$(\phi+1)\left(I - (1-b)\,\Gamma' - (1-\alpha)\,b\Theta'\right)^{-1}\overrightarrow{\log A} = \log C$$

Remember the equation for sales

$$\frac{\overrightarrow{S}'}{C} = \overrightarrow{\xi}'\left(I - (1-b)\,\Gamma' - (1-\alpha)\,b\Theta'\right)^{-1}$$

Thus

$$\log C = (\phi+1)\,\frac{\overrightarrow{S}'}{C}\overrightarrow{\log A} \qquad \text{and} \qquad \log L = \phi\frac{\overrightarrow{S}'}{C}\overrightarrow{\log A}$$

For version 1, you can do something similar.

Hulten (1978)

The Main Results

1. Aggregate productivity is a weighted average of productivity of the individual units:

$$A^{agg} \equiv \log \frac{C}{L} = \frac{\overrightarrow{S}'}{C} \overrightarrow{\log A}$$

The sum of the weights is bigger than 1.

2. Total output and labor inputs each depend on aggregate productivity and the labor supply elasticity

$$\log C = (\phi + 1)\, A^{agg} \text{ and } \log L = \phi A^{agg}$$

3. Combining (1) and (2)

$$\sigma_{\log C} = (\phi + 1) \frac{\sum S_j}{C} \left[\sum_i \left(\frac{S_i}{\sum S_j} \right)^2 \right]^{1/2} \sigma = \mu \cdot h \cdot \sigma,$$

where $\mu \equiv (\phi + 1) \cdot \sum \frac{S_i}{C}$

- Calibration: $h = 6\%$, $\sigma = 12\%$, $\mu = 2 \Rightarrow \sigma_{\log C} = 1.4\%$

Partial summary

- $h = 6\%$ and $\sigma = 12\%$ $\Rightarrow$ A calibration of a simple "islands" model implies that independent firm shocks can potentially meaningfully contribute to GDP volatility
- Rest of the paper:
  - Construct a measure of productivity shocks to individual firms
  - Regress GDP growth against productivity shocks of the largest firms.

Defining the granular residual

From before

$$\log \frac{Y_t}{Y_{t-1}} \propto \sum_i \frac{S_{i,t-1}}{Y_{it-1}} \log\left(\frac{A_{it}}{A_{i,t-1}}\right)$$

Define

$$\Gamma_t \equiv \sum_{i=1}^{100} \frac{S_{i,t-1}}{Y_{t-1}} \hat{\varepsilon}_{it},$$
$$\hat{\varepsilon}_{it} \equiv z_{it} - z_{i,t-1} - (\bar{z}_{It} - \bar{z}_{I,t-1})$$

where $z_{it} = \log\left(\frac{\text{sales of } i \text{ in year } t}{\text{employees of } i \text{ in year } t}\right)$, and $\bar{z}_{It}$ is the corresponding average labor productivity in firm i's industry, I.

On the granular residual

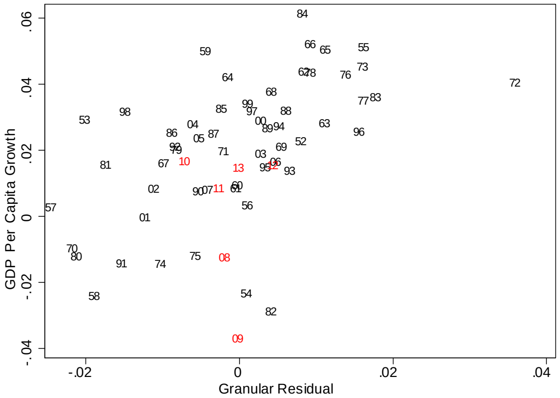
$\log\left(\frac{A_{it}}{A_{i,t-1}}\right)$ measures changes in TFP, not in labor productivity.

- For plants in the manufacturing sector these productivity measures have pretty different patterns (Syverson 2004):
  - 90-10 (75-25) difference of log labor productivity is roughly 1.4 (0.66)
  - 90-10 (75-25) difference of log TFP is 0.7 (0.29)

GDP growth and the granular residual

Sample	1952-2008			1952-2014		
Γ_t	2.8	2.9	3.7	2.9	2.9	3.9
Γ_{t-1}		3.1	3.4		3.1	3.4
Γ_{t-2}			2.1			2.3
Intercept	0.02	0.02	0.02	0.02	0.02	0.02
N	57	56	55	63	62	61
R^2	0.14	0.32	0.40	0.12	0.27	0.36
$\tilde{R}^2$	0.12	0.29	0.36	0.10	0.24	0.32

GDP growth and the granular residual



Granular events

- 1970 Strike at GM (labor productivity down 18%)
- 1972 Ford and Chrsyler have a rush of subcompact sales
- 1983 Launch of IBM PC (labor productivity up 10%)

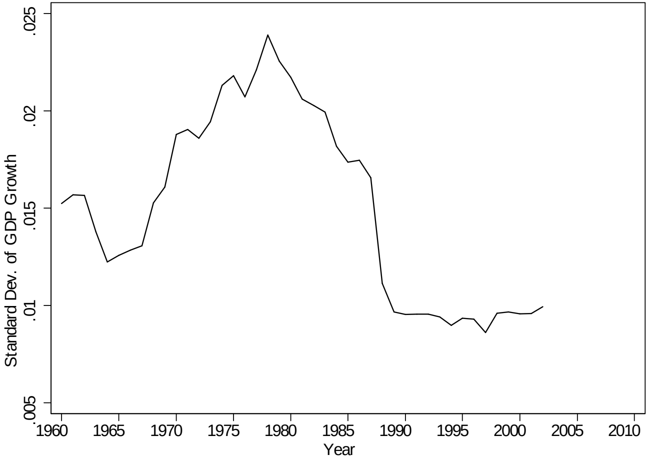
Predictive power of the granular residual

Intercept	0.015	0.019*	0.021*
Γ_{t-1}	3.5**		3.3**
Γ_{t-2}	1.2		2.3*
Monetary$_{t-1}$		-0.04	-0.05
Monetary$_{t-1}$		-0.02	0.04
Oil$_{t-1}$		$-8.7 \cdot 10^{-5}$	$-1.7 \cdot 10^{-4}$
Oil$_{t-2}$		$-6.9 \cdot 10^{-5}$	$-1.2 \cdot 10^{-4}$
3-month t-bill$_{t-1}$		-0.45	-0.41
3-month t-bill$_{t-2}$		0.43	0.39
Term Spread$_{t-1}$		0.38	0.40
Term Spread$_{t-2}$		0.27	-0.38
$\tilde{R}^2$	0.19	0.19	0.34

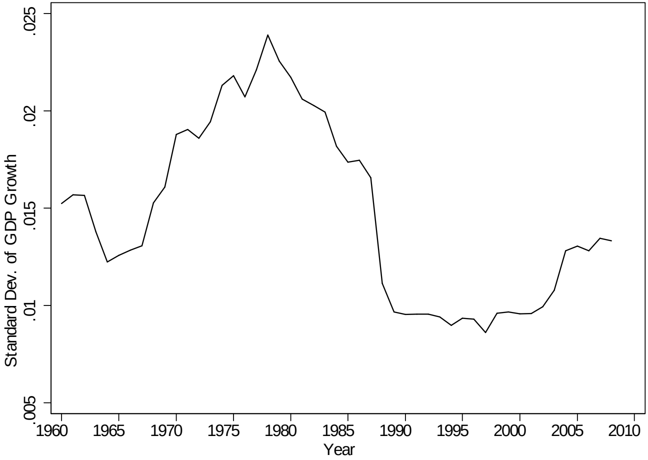
- Oil: (Hamilton 2003): current vs. last year's max oil price.
- Monetary policy shock: Residuals from FOMC decisions vs. FOMC forecasts (Romer and Romer 2004)
- Term spread: 5 year bond yield - 3 month bond yield.

Notes on Carvalho and Gabaix (2013): "The Great Diversification and its Undoing"

The Great Moderation



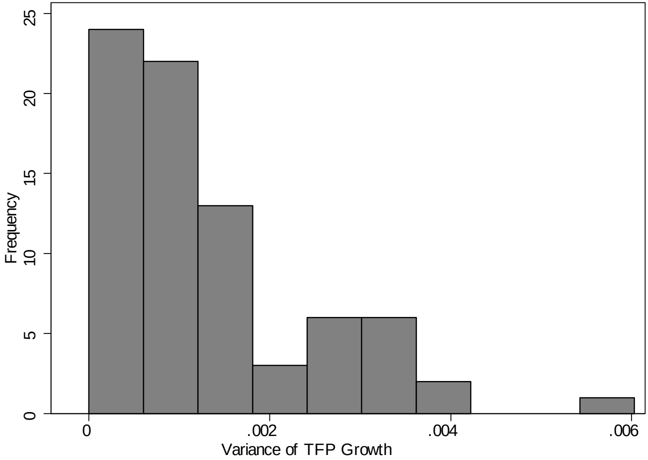
The Great Moderation and its Undoing



Motivation and question

- Why did volatility of GDP growth decrease beginning around 1980? And why did volatility increase beginning around 2005?
- Previous answers to the first question:
  - Stock and Watson (2003): It *doesn't* seem to have to do with better inventory management or more aggressive monetary policy.
  - Arias, Hansen, and Ohanian (2007): Aggregate TFP shocks have become less volatile)
  - Jaimovich and Siu (2009): Fewer young people (those with more elastic labor supply) in the work force
- New answer in this paper: Industry composition affects aggregate volatility.

Industries differ in their volatility



Fundamental Volatility

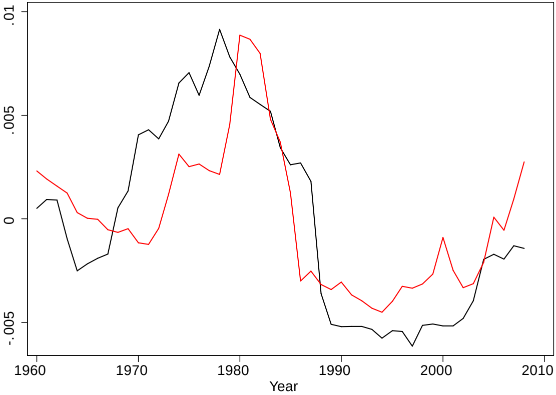
- Reminder from our islands economy
  - Suppose economy is made up of n units (firms or industries)
  - $\log GDP_t = \sum_i \frac{S_{it}}{GDP_t} \log(A_{it})$
  - $\log\left(\frac{GDP_{t+1}}{GDP_t}\right) \approx \sum_i \frac{S_{it}}{GDP_t} \log\left(\frac{A_{i,t+1}}{A_{it}}\right)$
- Suppose $\frac{A_{i,t+1}}{A_{it}}$ are i.i.d. across time and industries, with standard deviation σ_i

$$\mathrm{SD}\left[\log\left(\frac{GDP_{t+1}}{GDP_t}\right)\right] \approx \left[\sum_i \left(\frac{S_{it}}{GDP_t}\right)^{\frac{1}{2}} (\sigma_i)^2\right]^{1/2}$$

- Potentially
  - productivity shocks are correlated, have volatilities that change over time.
  - things besides industries' TFP change from one period to the next

Fundamental Volatility

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$



Outline

- Definitions and data sources
  - TFP in each industry
  - σ_{GDP}
- Fundamental volatility accounts for the break in GDP volatility.
- Sources of fundamental volatility
- Fundamental volatility and GDP volatility in other countries

Industry TFP Volatility

- KLEMS data from Dale Jorgenson (http://hdl.handle.net/1902.1/11155)
- For each industry×year define TFP growth as:

$$
\begin{aligned}
\Delta TFP_{it} = \log\left(\frac{y_{it+1}}{y_{it}}\right) \\
&- \frac{1}{2}\left(s^k_{it} + s^k_{it+1}\right)\log\left(\frac{k_{it+1}}{k_{it}}\right) \\
&- \frac{1}{2}\left(s^l_{it} + s^l_{it+1}\right)\log\left(\frac{l_{it+1}}{l_{it}}\right) \\
&- \frac{1}{2}\left(s^m_{it} + s^m_{it+1}\right)\log\left(\frac{m_{it+1}}{m_{it}}\right)
\end{aligned}
$$

  where s^l_{it} (s^m_{it}, s^k_{it}) is industry i's cost share of labor (intermediate inputs, capital) at time t.
- $\sigma_i \equiv \text{SD}(\Delta TFP_{it})$

GDP Volatility

Three measures:

1) Rolling standard deviation

$$\sigma_t^{roll} = \text{SD}\left(y_{t-10}^{HP}, \ldots y_{t+10}^{HP}\right), \text{ where}$$

y_t^{HP} is deviation of log GDP from trend

2) Instantaneous standard deviation

$$\Delta y_s = \psi + \phi \Delta y_{s-1} + \epsilon_s$$

$$\sigma_t^{Inst} \equiv \frac{1}{2}\sqrt{\frac{\pi}{2}} \sum_{q=1}^{4} |\hat{\epsilon}_{t,q}|$$

3) σ_t^{HP} is the HP smoothed version of σ_t^{Inst}

Fundamental Volatility accounts for the break in GDP volatility.

$$LR_T = \frac{\prod_{t=1960}^{T} f_1(\eta_t) \prod_{T+1}^{2008} f_2(\eta_t)}{\prod_{t=1960}^{2008} f_0(\eta_t)}$$

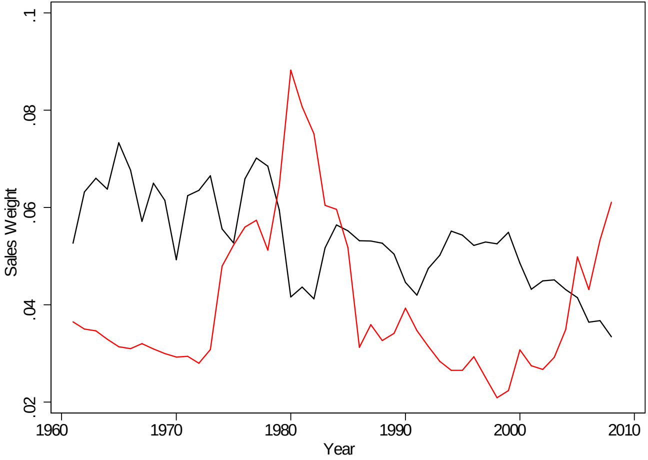
	$\sigma_{Yt}^{inst} = a + \eta_t$		$\sigma_{Yt}^{inst} = a + b\sigma_{Ft} + \eta_t$	
H_0	No break in a		No break in b	No break in a or b
$\max_T LR_T$	26.50	8.32	8.64	8.91
Reject null?	Yes	No	No	No
Estimated break date	1983	NA	NA	NA

Fundamental Volatility

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$

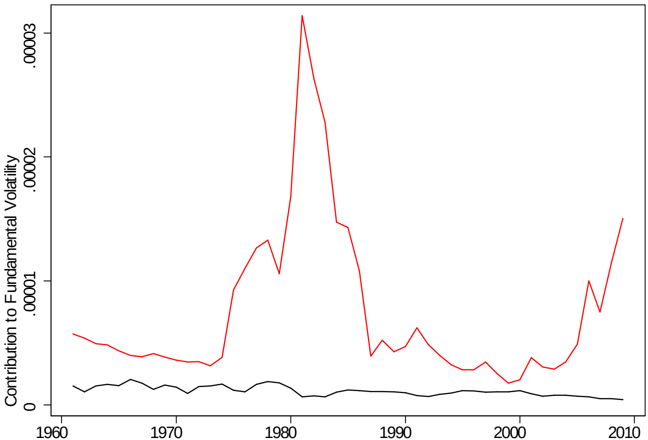
Sales weights: motor vehicles and petroleum

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$



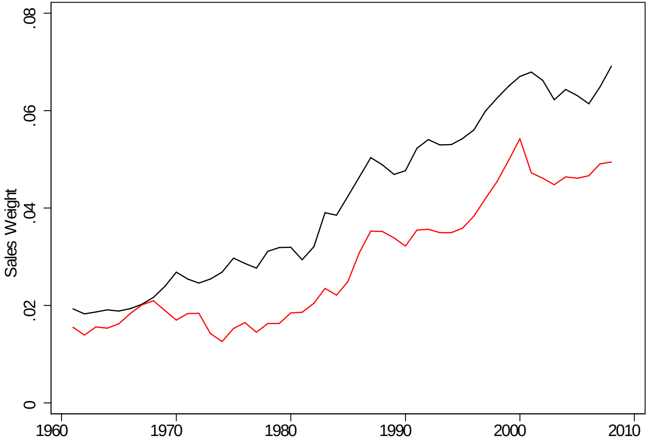
Contribution to fundamental volatility: motor vehicles and petroleum

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$



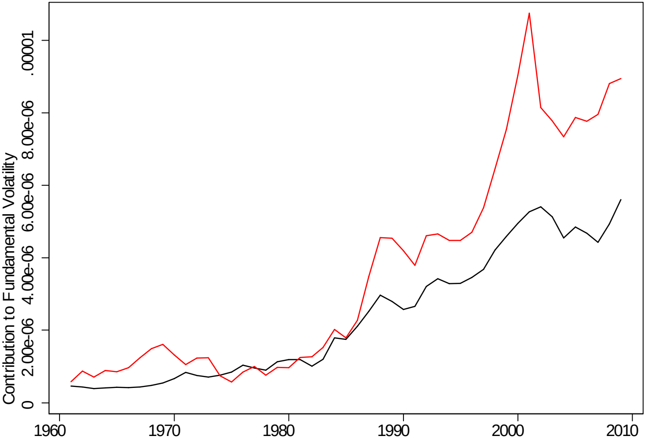
Sales weights: depository and nondepository financial institutions

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$

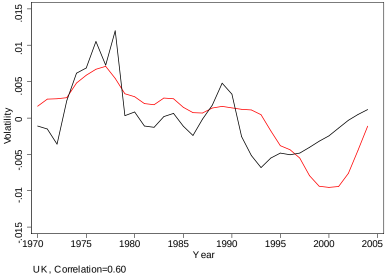


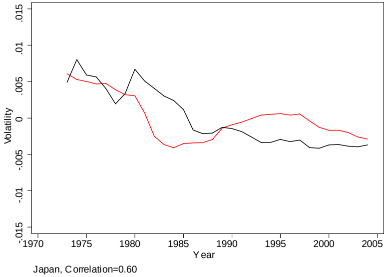
Contribution to fundamental volatility: depository and nondepository financial institutions

$$\sigma_{Ft} = \left[\sum_i \left(\frac{S_{it}}{GDP_t} \right)^2 (\sigma_i)^2 \right]^{1/2}$$

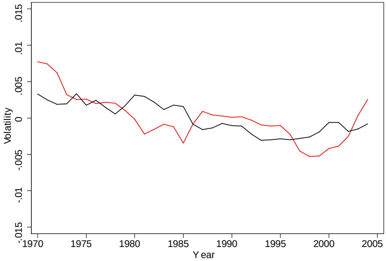


Fundamental volatility tracks GDP volatility in other countries

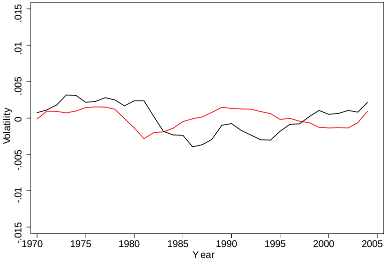




Fundamental volatility tracks GDP volatility in other countries



Germany, Correlation=0.54



France, Correlation=0.03

Conclusion

Summary:

- GDP volatility changes over time.
- Volatility changes reflect changes in the importance of different types of firms in the economy.
  - Implies that firm/industry-level shocks are important for aggregate volatility.

Next steps:

- To what extent are the economy's shocks independent across firms (or industries)?

Notes on Long and Plosser (1983): "Real Business Cycles"

Model

- From the board:

$$X_{j\to i,t} = \beta \frac{\gamma_i}{\gamma_j} a_{j\to i} Y_{jt}$$

$$L_{it} = \frac{\beta \gamma_i b_i}{\theta_0 + \beta \sum \gamma_j b_j} H$$

- Plug these expressions into the production function:

$$\begin{aligned} Y_{i,t+1} &= \lambda_{i,t+1} \left(L_{it}\right)^{b_i} \prod_j \left(X_{j\to i,t}\right)^{a_{j\to i}} \\ &= \lambda_{i,t+1} \underbrace{\left(\frac{\beta \gamma_i b_i}{\theta_0 + \beta \sum \gamma_j b_j} H\right)^{b_i} \prod_j \left(\beta \frac{\gamma_i}{\gamma_j} a_{j\to i}\right)^{a_{j\to i}}}_{\equiv \exp\{k_i\}} \prod_j \left(Y_{jt}\right)^{a_{j\to i}} \end{aligned}$$

$$\underbrace{\log Y_{i,t+1}}_{\equiv y_{i,t+1}} = \underbrace{\log\left(\lambda_{i,t+1}\right)}_{\equiv \eta_{t+1}} + k_i + \sum_j a_{j\to i} \log Y_{jt}$$

Theoretical predictions

- Writing the previous equation in vector form:

$$y_{t+1} = A' y_t + k + \eta_{t+1}$$

- Define $\tilde{y}_{t+1} \equiv y_{t+1} - (I - A')^{-1} k$

$$\tilde{y}_{t+1} = A' \tilde{y}_t + \eta_{t+1}$$

$$\tilde{y}_t = \sum_{j=0}^{\infty} (A')^j \eta_{t-j}$$

- Assume $\text{Var}(\eta_t) = I$. Then:

$$\text{Var}(\tilde{y}_t) = \sum_{j=0}^{\infty} (A')^j A^j$$

$$\text{Cov}(\tilde{y}_t, \tilde{y}_{t-1}) = \sum_{j=0}^{\infty} (A')^j A^{j+1}$$

Data

- BEA: 1992 Input/Output Table & Capital Flows Table.
- Dale Jorgenson: Annual data on industries' production.
  - Agriculture (5%)
  - Construction (6%)
  - Durable Manufacturing (16%)
  - Nondurable Manufacturing (16%)
  - Transportation (10%) Wholesale/Retail (14%)
  - Finance, Insurance, and Real Estate (13%)
  - Personal and Business Services (20%).

Intermediate Input and Capital Flows

Agriculture	0.25	0.01	0.01	0.09	0.03	0.00	0.0	0.00
Construction	0.04	0.02	0.02	0.03	0.10	0.04	0.32	0.03
Durable Manf.	0.12	0.16	0.40	0.10	0.17	0.08	0.04	0.08
Nondurable	0.08	0.03	0.04	0.31	0.04	0.04	0.01	0.08
Transport	0.05	0.02	0.04	0.07	0.20	0.06	0.03	0.03
Whole/Retail	0.07	0.28	0.08	0.08	0.04	0.09	0.02	0.03
FIRE	0.11	0.03	0.02	0.03	0.05	0.10	0.26	0.06
Other Services	0.04	0.06	0.07	0.07	0.10	0.12	0.08	0.15
Capital + Materials Share	0.78	0.60	0.69	0.77	0.73	0.52	0.76	0.46

Theoretical Predictions

Correlations and autocorrelations

Agriculture	1	0.09	0.11	0.13	0.11	0.08	0.10	0.07	0.31
Construction		1	0.12	0.09	0.09	0.07	0.06	0.06	0.07
Durable Manf.			1	0.11	0.13	0.08	0.08	0.08	0.42
Nondurable				1	0.11	0.08	0.08	0.08	0.36
Transport					1	0.08	0.11	0.07	0.26
Whole/Retail						1	0.08	0.06	0.12
FIRE							1	0.07	0.30
Other Services								1	0.17

Average correlation: 0.20
Average autocorrelation: 0.23

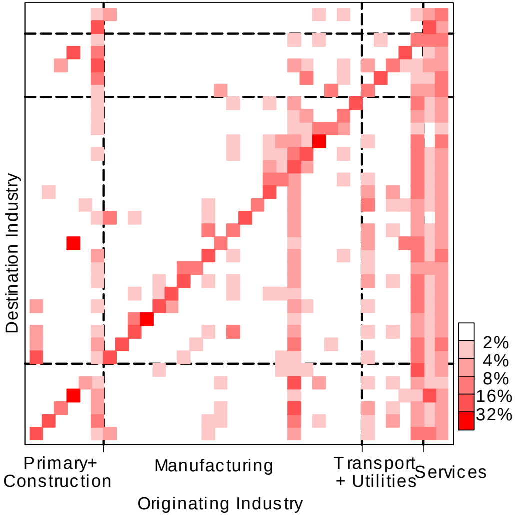
Empirical counterparts

Correlations and autocorrelations:

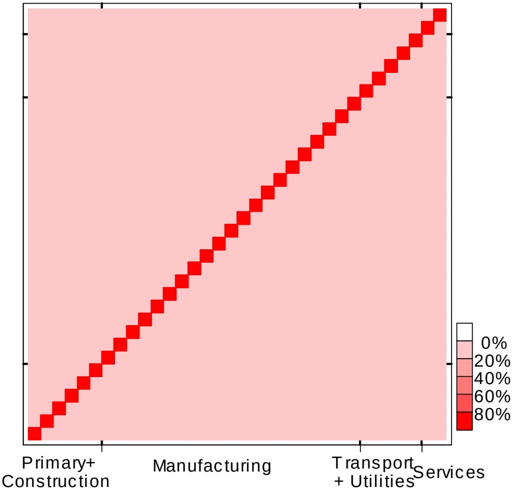
Agriculture	1	0.34	-0.09	0.55	-0.16	0.08	0.10	0.07	0.34
Construction		1	0.43	0.28	0.78	0.43	-0.13	0.05	0.25
Durable			1	0.00	0.59	0.03	0.01	0.68	-0.02
Nondurable				1	0.79	0.06	-0.41	-0.05	0.21
Transport					1	0.41	-0.46	0.14	0.30
Whole/Retail						1	0.22	-0.09	-0.07
FIRE							1	-0.04	0.53
Other Serv.								1	0.28

Average correlation: 0.28 (theoretical=0.20)
Average autocorrelation: 0.25 (theoretical=0.23)

Intermediate Input and Capital Flows



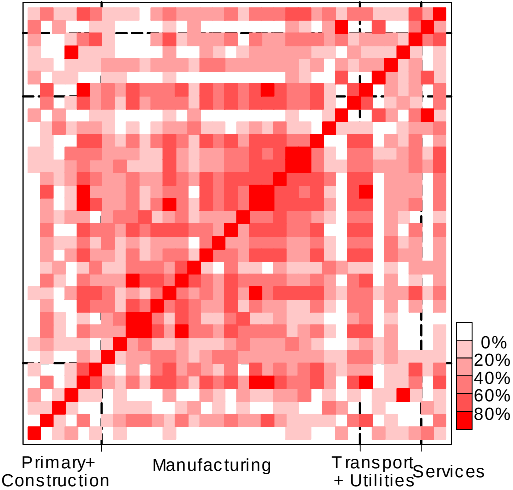
Theoretical Prediction



Average correlation: 7%

Average autocorrelation: 22%

Empirical counterpart



Average correlation: 26% (theoretical=0.07)
Average autocorrelation: 64% (theoretical=0.22)

Conclusion

- Two common characteristics of business cycles: co-movement and persistence
- Input-output linkages can spread industry-specific shocks over time, across industries.
- With data on many industries, the amplification seems not to be "strong enough"$\Rightarrow$ residual cross-industry correlation of productivity shocks.
- $\Delta \tilde{y}_{t+1} - A' \Delta \tilde{y}_t = \log \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \Rightarrow$ Can use data on $\Delta \tilde{y}_{t+1} - A' \Delta \tilde{y}_t$ to infer correlation of productivity shocks.

Notes on Foerser, Sarte, Watson (2011) "Sectoral vs. Aggregate Shocks: A Structural Factor Analysis of Industrial Production"

Research questions

1. How correlated are shocks to industries' productivities?
2. What fraction of industrial production volatility is due to common shocks? Industry-specific shocks?
3. Are the answers to (1) and (2) different for different points in the sample? Were common shocks or industry-specific shocks less volatile during the Great Moderation (after 1983)?

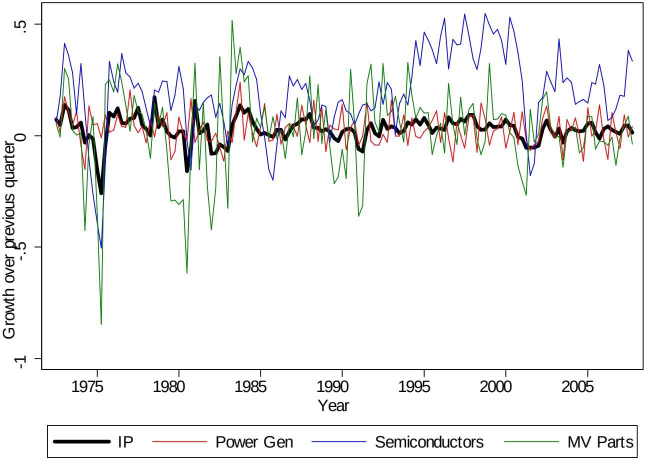
Outline

- Data
- Statistical factor analysis
- Model and structural factor analysis

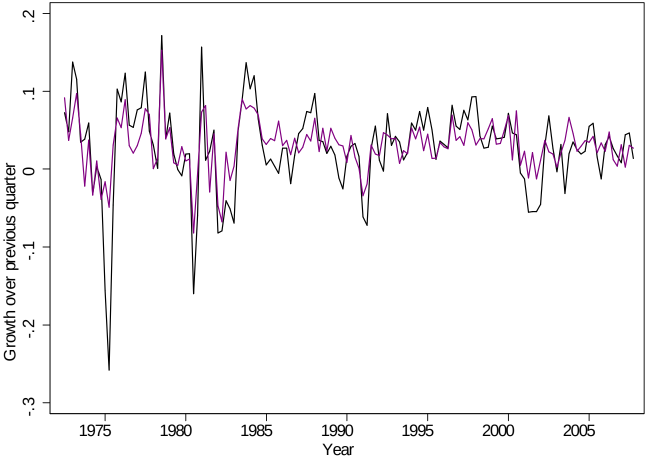
Data

- BEA: Input/Output Table & Capital Flow Table, from 1997.
- Federal Reserve Board: Quarterly data on industrial production, from 1972 to 2011.
  - Quarterly data
  - 117 industries in manufacturing, mining, energy, and publishing.

Industrial production and its components



Industrial production tracks GDP



Principal component analysis

- Define g_t as the vector of sectoral growth rates, $\log\left(\frac{Y_{t+1}}{Y_t}\right)$, and w_t as the weight of each industry within the industrial sector.
- How can we best measure the fraction of variation in $w_t' \cdot g_t$ that is due to "common shocks"?
- Suppose that

$$\underset{117\times1}{g_t} = \underset{117\times2}{\Lambda} \cdot \underset{2\times1}{F_t} + \underset{117\times1}{u_t}$$

  where F_t is some small (e.g., two) number of common factors, and u_t are idiosyncratic shocks (the covariance matrix of u has zero off-diagonal terms), and F_t and u_t are uncorrelated.
- Use principal component analysis to choose Λ, F_t so that ΛF explains the maximum possible variance of the g_t vector. These columns of F will represent the common shocks.

Principal component analysis

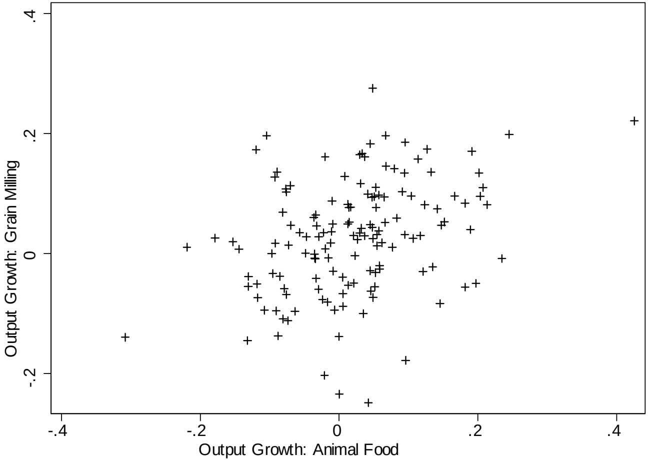
- From the last slide

$$g_t = \Lambda \cdot F_t + u_t$$

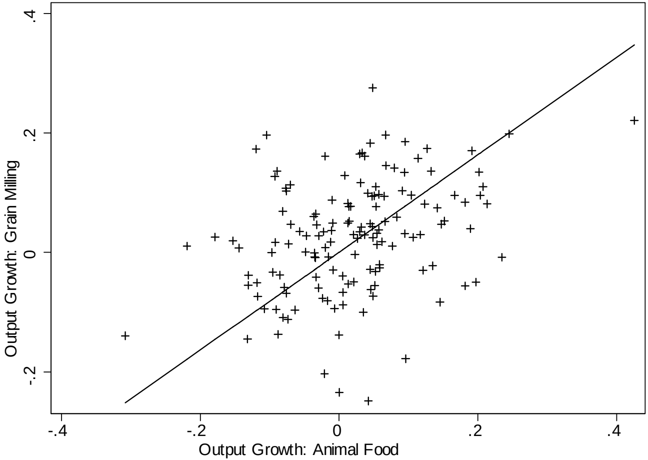
- Note that Λ and F_t are not separately identified. $\Lambda F_t = \underbrace{\Lambda\vartheta}_{\tilde{\Lambda}}\underbrace{\vartheta^{-1}F_t}_{\tilde{F}_t}$. We will normalize Λ so that the lengths of each column equal 1.
- Useful formulas:

$$\begin{aligned} \Sigma_{gg} &= \Lambda\Sigma_{FF}\Lambda' + \Sigma_{uu} \\ \sigma_g^2 &= \bar{w}'\Lambda\Sigma_{FF}\Lambda'\bar{w} + \bar{w}'\Sigma_{uu}\bar{w} \\ R^2(F) &= \frac{\bar{w}'\Lambda\Sigma_{FF}\Lambda'\bar{w}}{\sigma_g^2} \end{aligned}$$

Principal component analysis: 2-d to 1-d



Principal component analysis: 2-d to 1-d



Principal component analysis

The idea is to find the linear combination of the data that explains the greatest possible variance

$$\max_{\|\Lambda_1\|=1} \Lambda_1' \Sigma_{\text{Animal,Grain}} \Lambda_1$$
$$= \max_{\Lambda_1} \Lambda_1' \Sigma_{\text{Animal,Grain}} \Lambda_1 + \mu_1 \left(1 - \Lambda_1' \Lambda_1\right)$$

First order conditions:

$$2\Sigma_{\text{Animal,Grain}} \Lambda_1 = 2\mu_1 \Lambda_1$$
$$\Sigma_{\text{Animal,Grain}} \Lambda_1 = \mu_1 \Lambda_1$$

Note that

$$\Lambda_1' \Sigma_{\text{Animal,Grain}} \Lambda_1 = \Lambda_1' \mu_1 \Lambda_1 = \mu_1$$

To maximize the left hand side, choose the unit-length eigenvector associated with the largest eigenvalue of $\Sigma_{\text{Animal,Grain}}$.

Principal component analysis

Can extend this idea to many (say 117) data series and multiple (say 2) factors.

Suppose we have 117 data series and we have computed the first factor $F_1 = g' \cdot \Lambda_1$. The problem is now to find a vector Λ_2 that is orthogonal to Λ_1 and explains the greatest possible variance:

$$\max_{\Lambda_2} \Lambda_2' \Sigma_{gg} \Lambda_2 + \mu_2 \left(1 - \Lambda_2' \Lambda_2\right) + \kappa \Lambda_2' \Lambda_1$$

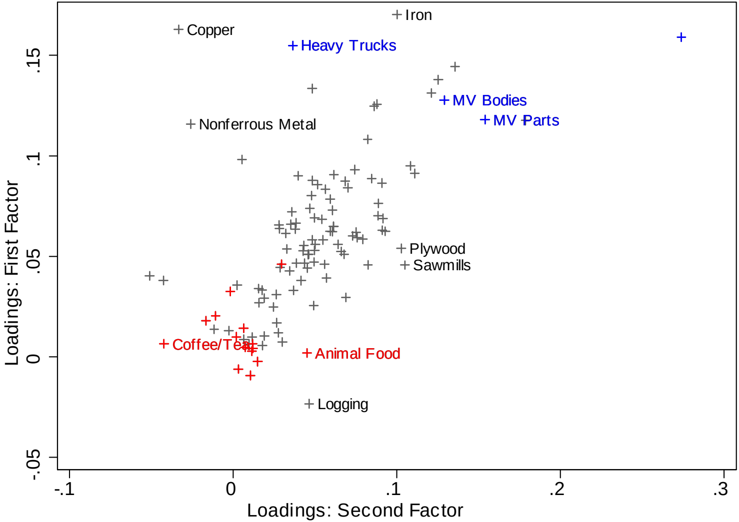
First order conditions:

$$\Sigma_{gg} \Lambda_2 = \mu_2 \Lambda_2$$

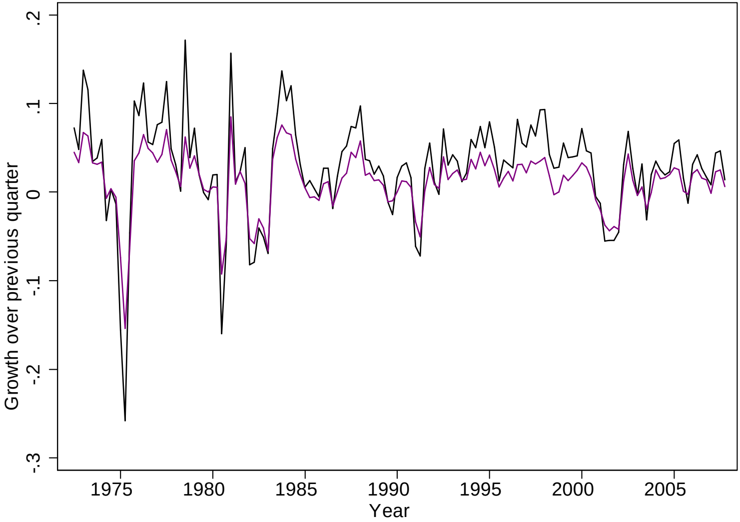
The solution to this maximization problem, Λ_2 will be the eigenvector associated with the second largest eigenvalue of Σ_{gg}

Side note: See Bai and Ng (2003) on how to choose the number of factors (similar to minimizng Mallows's C_p)

The two columns of Λ



Industrial production and its factor component



$R^2(F) \approx 0.9$

Partial summary

- The story so far: there is a strong common component to industrial production.
- Is this because there are common shocks? Or because there are independent shocks transmitted via input-output relationships?
- Rest of the paper: Use a model with input-output linkages to back out productivity shocks for each industry-quarter. Perform factor analysis on the productivity shocks.
  - N perfectly competitive sectors, which produce using capital, labor, and the output of other sectors.
  - Consumers have preferences over leisure and consumption of the goods produced by the N industries.
  - Productivity growth is distributed $\mathcal{N}(0, \Sigma_{\omega\omega})$; ω will admit an factor representation.

Model: Market Clearing

- Output can be used for consumption, as an intermediate input, or to increase one of the N capital stocks:

$$Y_{tj} = C_{tj} + \sum_{i=1}^{N} M_{t,j \to i} + \sum_{i=1}^{N} X_{t,j \to i} \quad \forall j \in \{1, ... N\}$$

Model: Preferences

- Consumers' lifetime utilities are given by:

$$U = \sum_{t=0}^{\infty} \beta^t \left[\sum_{i=1}^{N} \frac{(C_{ti})^{1-\sigma} - 1}{1-\sigma} - \psi L_{ti} \right]$$

- ψ: disutility from work
- σ: preference elasticity of substitution, intertemporal elasticity of substitution.

Model: Production

- The production technology of each sector is given by:

$$Y_{tj} = A_{tj} \left(K_{tj}\right)^{\alpha_j} M_{tj} \left(L_{tj}\right)^{1-\alpha_j-\sum_i \gamma_{ij}}$$

- The intermediate input bundle of sector j consists of the purchases from the other sectors:

$$M_{tj} = \prod_i X_{t,i\to j}^{\gamma_{ij}}$$

- γ_{ij} is the share of good i used in the production of the good-j intermediate input.

Model: Evolution of Capital, Productivity

- Industry-j-specific capital evolves according to:

$$K_{t+1,j} = (1-\delta) K_{tj} + Z_{tj}$$
$$\text{where } Z_{tj} = \prod_i X_{t,i\to j}^{\theta_{ij}}$$

- θ_{ij} is the share of good i used in the production of the good-j capital input.

- Productivity in each sector evolves according to a random walk:

$$\log A_{t+1,j} = \log A_{tj} + \omega_{t+1,j}, \qquad \omega_{t+1} \sim \mathcal{N}(0, \Sigma_{\omega\omega})$$

Solution

- The model yields the following (log-linear-approximate) expression for the evolution of output:

$$\begin{pmatrix} \Delta \log Y_{t+1,1} \\ \Delta \log Y_{t+1,2} \\ \dots. \end{pmatrix} = Q \begin{pmatrix} \Delta_{t,1} \\ \Delta_{t,2} \\ \dots. \end{pmatrix} + R \begin{pmatrix} \omega_{t1} \\ \omega_{t2} \\ \dots \end{pmatrix} + S \begin{pmatrix} \omega_{t+1,1} \\ \omega_{t+1,2} \\ \dots \end{pmatrix}$$

- Q, R, and S are specified, given $(\beta, \delta, \psi, \sigma, \alpha_i, \theta_{ij}, \gamma_{ij})$
- Given these parameters, one can back out innovations to productivity (setting $\omega_0 = 0$):

$$\begin{pmatrix} \omega_{t+1,1} \\ \omega_{t+1,2} \\ \dots \end{pmatrix} = S^{-1} \left[\begin{pmatrix} \Delta \log Y_{t+1,1} \\ \Delta \log Y_{t+1,2} \\ \dots. \end{pmatrix} - Q \begin{pmatrix} \Delta \log Y_{t1} \\ \Delta \log Y_{t2} \\ \dots. \end{pmatrix} - R \begin{pmatrix} \omega_{t1} \\ \omega_{t2} \\ \dots \end{pmatrix} \right]$$

- Next step: Calibrate model's parameters, which will allow us to back out ω_{tj}.

Calibration

Parameter	Value/Source
β-discount factor	0.99
δ-capital depreciation rate	0.025
ψ-disutility from work	1
σ-consumers' elasticity, across goods	1
α_i-capital share in production of i	1997 BEA I.O. Tables
$1 - \alpha_i - \sum_j \gamma_{ij}$-share of capital/labor in production of i	1997 BEA I.O. Tables
γ_{ij}-share of good i in production of j's intermediate input	1997 BEA I.O. Tables
θ_{ij}-share of good i in production of j's capital input	1998 Capital Flow Tables

Calibration of $\Sigma_{\omega\omega}$

- Given the other parameters, we know ω_{tj} for all time periods industries.
- Two calibrations:
  - $\Sigma_{\omega\omega}$ is diagonal, with the j, j^{th} entry equal to the sample variance of ω_{tj}
  - Perform principal component analysis on the ω_{tj}: $\Sigma_{\omega\omega} = \Lambda_S \Sigma_{SS} \Lambda_S + \Sigma_{uu}$, with a 2-dim. common factor, S_t
- With $\Sigma_{\omega\omega}$ in hand, we compute the following statistics:
  - $\bar{\rho}_{ij}$: average correlation in the growth rates for two industries
  - σ_g: standard deviation of the growth rate of industrial production
  - $R^2(S)$: fraction of the variation in industrial production growth explained by the common factors.

Results

	Period	$\bar{\rho}_{ij}$	σ_g	$R^2\,(S)$
Data	72-83	0.27	8.8	
	84-07	0.11	3.6	
Uncorrelated Shocks	72-83	0.05	5.1	
	84-07	0.04	3.1	
2 Common Factors	72-83	0.26	9.5	0.81
	84-07	0.10	4.1	0.50

Comparison to other models

- Foerster et al.:

$$g_{t+1} = Qg_t + S\omega_{t+1} - R\omega_t$$

- Long and Plosser: Materials arrive with a one-period lag, no capital, log preferences for consumption and leisure.

$$g_{t+1} = \Gamma' g_t + \omega_{t+1}$$
$$\Sigma_{gg} = \sum_{i=0}^{\infty} \left(\Gamma'\right)^i \Sigma_{\omega\omega} \Gamma'$$

- Carvalho (2007), Acemoglu et al (2012): No capital, log preferences for consumption, perfectly elastic labor supply

$$g_{t+1} = \left(I - \Gamma'\right) \omega_{t+1}$$
$$\Sigma_{gg} = \left(I - \Gamma'\right) \Sigma_{\omega\omega} \left(I - \Gamma\right)$$

Model performance with independent productivity shocks

	$\bar{\rho}_{ij}$	σ_g	σ_g(diag)	σ_g(scaled)$\div$ $\sigma_{g,bench}$(scaled)
Data	0.19	5.80	1.85	
Benchmark	0.04	3.87	1.88	1.00
Long-Plosser	0.01	2.66	2.07	0.39
Carvalho	0.04	3.15	1.64	0.87
Benchmark, $\theta = I$	0.02	3.86	2.43	0.59
Benchmark, $\Sigma_{\omega\omega} = \sigma^2 I$	0.04	5.72	2.99	0.86
Benchmark, Γ, α: average	0.05	3.30	1.71	0.87

σ_g(scaled) is defined as the σ_g computed in an alternative calibration in which $\Sigma_{\omega\omega}$ is chosen so that "model-implied variance of IP growth associated with the diagonal elements of Σ_{gg} correspond to the value in the data."

Do the industry definitions matter?

	Period	2-digit 26 inds.	3-digit 88 inds.	4-digit 117 inds.
Data $\bar{\rho}_{ij}$	72-83	0.38	0.29	0.27
	84-07	0.22	0.13	0.11
Independent Error $\bar{\rho}_{ij}$	72-83	0.09	0.05	0.05
	84-07	0.07	0.05	0.04
R^2 (S)	72-83	0.76	0.85	0.81
	84-07	0.53	0.53	0.50

Conclusion

- Summary:
  - Industry-specific shocks explain about 40% of the variation in industrial production
  - Lower (20%) in the pre-Great Moderation period; higher (50%) in the Great Moderation. Common shocks became less volatile during the great moderation
- Extensions:
  - Apply this model to the whole economy, not just the goods-producing sectors (Ando 2014)
  - Decompose output variation into firm-specific, industry-specific, and common shocks.