Notes on Kongsamut et al. (2001)

The goal of this model is to be consistent with the Kaldor facts (constancy of growth rates, capital shares, capital-output ratios) and the "Kuznets facts" (employment in services is increasing, in manufacturing it is roughly constant, in agriculture it is decreasing). The second fact that the authors advance is that the real consumption of different products vary a lot less over time than nominal consumption shares. Related, employment growth and relative price inflation are positively correlated (services become relatively more expensive and have a higher and higher worker share; opposite for agriculture). The primary mechanism that the authors propose is that there are varying income elasticities for food, manufactured products, and services.

There are three sectors: agriculture, services, and manufacturing; labor can be reallocated, within periods, without adjustment costs. This makes sense because we are trying to match long-run patterns.

Manufactured products can either be consumed or can be used to as investment, to augment the next period capital stock. The production of agricultural products and services are straightforward

$$\begin{aligned} A_t &= B_A F(\phi_t^A K_t, N_t^A X_t) \\ S_t &= B_S F(\phi_t^S K_t, N_t^S X_t) \\ M_t + \delta K + \dot{K} &= B_M F(\phi_t^M K_t, N_t^M X_t) \end{aligned}$$

The total stock of labor is equal to 1 (i.e., the N's sum up to 1). Same for the ϕ's; they represent the fraction of capital that is used in each of its three possible uses.

The final building block for the production side of the economy, g is the constant growth rate of the labor-augmenting technology term. That is: $\dot{X}_t = X_t \cdot g$. It is crucial that the productivity growth is labor augmenting (and not, for example, factor neutral).

We will set the manufactured good to be the numeraire; call P_A and P_S the price of the agricultural good and the service good (these can potentially vary across periods, but they won't so long as the productivity growth rates are constant across industries.

There are other restrictions: that productivity growth is the same for all industries, and that the production functions of the three sectors are the same. The authors show, in a working paper version of the article, that these restrictions can be relaxed.

The differential growth rates will result from the specification on preferences that we make. (It can't be from the technology side, since all industries have identical production functions and productivity growth rates.)

$$
\begin{aligned}
U &= \int e^{-\rho t}\frac{C(t)^{1-\sigma}-1}{1-\sigma}dt, \text{ where} \\
C(t) &= \left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta}
\end{aligned}
$$

The key, here, is that the income elasticities of these products are different from 1:

$$
\begin{aligned}
\max \left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta} \text{ s.t. } I(t) &= P_S S(t) + P_A A(t) + M(t) \\
S(t)+\bar{S} &= \frac{\theta}{P_S}\left(I + P_S\bar{S} - P_A\bar{A}\right) \\
A(t)-\bar{A} &= \frac{\beta}{P_A}\left(I + P_S\bar{S} - P_A\bar{A}\right) \\
M(t) &= \gamma\left(I + P_S\bar{S} - P_A\bar{A}\right)
\end{aligned}
$$

This is very similar to the Cobb-Douglas demand curves; the "effective income" is shifted down/up by $-P_S\bar{S}+P_A\bar{A}$, and there is some "necessary"/ "surplus" amount of the agricultural/service good that the representative consumer purchases before getting to the Cobb-Douglas part.

$$
\begin{aligned}
\frac{\partial \log S(t)}{\partial \log I} &= \frac{I\theta}{(I+P_S\bar{S}-P_A\bar{A})\theta - P_S\bar{S}} > 1 \\
\frac{\partial \log A(t)}{\partial \log I} &= \frac{I\beta}{(I+P_S\bar{S}-P_A\bar{A})\beta + P_A\bar{A}} < 1
\end{aligned}
$$

The goal, remember, is to provide conditions under which the aggregate economy behaves as if it is on a balanced growth path. (Industries will grow at different rates, agriculture will grow more slowly; services more quickly. These rates will change over time as well. But we want output, capital per worker, and other aggregate variables to grow at constant, related rates).

The first step will be to write out the real interest rate, and find conditions under which this real interest rate in constant. Then we will solve for the growth rate of consumption shares and labor shares of each of the sector outputs, and then finally graphically analyze the transitional dynamics when the condition for constant real interest rates fails to hold.

We can write out the real interest rate using a no-arbitrage-type condition. Consumers can buy a unit of capital today in exchange for one unit of the manufactured good, rent that unit to firms who will pay $B_M F_1(k,1)$ for it, and have $(1-\delta)$ units of capital in the following period from which they will earn $(1-\delta)$ units of the manufactured good. They would be indifferent with this series of transactions and buying a bond that earned a gross interest rate of $(1+r)$. (One fuzzy thing about these statements is that the model is written in continuous time.) Putting these statements together:

$$\begin{aligned} 1+r &= B_M F_1(k,1) + 1 - \delta \\ r &= B_M F_1(k,1) - \delta \end{aligned}$$

The real interest rate will be constant if k is constant, which occurs if and only if $\frac{K}{X}$ is constant (meaning that K must grow at rate g).

The fact that all three products are produced using the same production function F implies that the allocation of capital to labor is the same across industries. Since the ϕ's and N's each sum to 1:

$$\frac{\phi_t^A}{N_t^A} = \frac{\phi_t^S}{N_t^S} = \frac{\phi_t^M}{N_t^M}$$

The budget constraint:[1]

$$\dot{K}_t + \delta K_t + M_t + P_A A_t - P_S S_t = X \cdot B_M F(k,1)$$

Along a BGP, the left hand side grows at g; the left hand side grows at rate g if and only if

[1] Use the budget constraint for the manufactured good, then the homogeneity of the function F, and the fact that $\frac{\phi_t^A}{N_t^A} = \frac{\phi_t^S}{N_t^S} = \frac{\phi_t^M}{N_t^M} = 1$:

$$\begin{aligned} \dot{K}_t + \delta K_t + M_t &= B_M F\left(\phi_t^M K_t, N_t^M X_t\right) \\ &= N_t^M X_t B_M F\left(\frac{\phi_t^M K_t}{N_t^M X_t}, 1\right) \\ &= N_t^M X_t B_M F\left(\frac{K_t}{X_t}, 1\right) \\ &= N_t^M X_t B_M F\left(k_t, 1\right) \end{aligned}$$

Add services and agricultural goods to the previous equation:

$$\begin{aligned} \dot{K}_t + \delta K_t + M_t + N_t^S X_t B_M F(k_t,1) + N_t^A X_t B_M F(k_t,1) &= N_t^M X_t B_M F(k_t,1) + N_t^S X_t B_M F(k_t,1) \\ &\quad + N_t^A X_t B_M F(k_t,1) \\ &= X_t B_M F(k_t,1) \end{aligned}$$

Now use the fact that $B_M = B_A P_A$ and $B_M = B_S P_S$ and then the definition of S and A:

$$\begin{aligned} \dot{K}_t + \delta K_t + M_t + N_t^S X_t B_S P_S F(k_t,1) + N_t^A X_t B_A P_A F(k_t,1) &= N_t^M X_t B_M F(k_t,1) + N_t^S X_t B_M F(k_t,1) \\ &\quad + N_t^A X_t B_M F(k_t,1) \\ \dot{K}_t + \delta K_t + M_t + P_A A_t + P_S S_t &= X_t B_M F(k_t,1) \end{aligned}$$

$$P_A A_t - P_S S_t \text{ grows at rate } g$$

From before:

$$\begin{aligned} P_S S_t &= -P_S \bar{S} + \theta \left(I + P_S \bar{S} - P_A \bar{A} \right) \\ P_A A_t &= P_A \bar{A} + \beta \left(I + P_S \bar{S} - P_A \bar{A} \right) \end{aligned}$$

Thus we need $-P_S \bar{S} + P_A \bar{A} + (\theta + \beta) \left(I + P_S \bar{S} - P_A \bar{A} \right)$ to grow at rate g. This can only happen if $P_S \bar{S} = P_A \bar{A}$, which is equivalent to $\bar{A} B_S = \bar{S} B_A$. This is the main proposition of the paper.[2]

What are the growth rates of A_t, M_t, S_t?

$$\begin{aligned} \left(S_t + \bar{S} \right) &= \left(S_0 + \bar{S} \right) e^{gt} \\ \dot{S}_t &= \frac{\partial S_t}{\partial t} = \frac{\partial \left(S_0 e^{gt} + \bar{S} \left(e^{gt} - 1 \right) \right)}{\partial t} = g \left(S_0 + \bar{S} \right) e^{gt} \\ \frac{\dot{S}_t}{S_t} &= g \frac{S_t + \bar{S}}{S_t} \end{aligned}$$

Similarly:

$$\frac{\dot{A}_t}{A_t} = g \frac{A_t - \bar{A}}{A_t}$$

What about labor shares? From the definition of the production function

$$\begin{aligned} N_t^A &= \frac{A_t}{B_A F(k,1) X_t} \\ \dot{N}_t^A &= \frac{\dot{A}_t}{B_A F(k,1) X_t} - \frac{A_t \cdot \dot{X}_t}{B_A F(k,1) \left(X_t \right)^2} \\ &= \frac{g \left(A_t - \bar{A} \right)}{B_A F(k,1) X_t} - \frac{A_t \cdot g}{B_A F(k,1) X_t} \\ &= -\frac{g \bar{A}}{B_A F(k,1) X_t} \text{ ; (goes to 0 as } t \to \infty) \end{aligned}$$

[2] Side note: Is this a reasonable condition? In the data, the relative price for services (vs. agricultural products) has been trending upward. Kongsamut et al. would counter that quality improvements have led conventional service-price indices to have overstated growth rates. Still though, this knife edge condition will almost surely be violated in the data. At the same time, the requirement of constant r is probably too strict. It could be the case that $P_S \bar{S} \approx P_A \bar{A}$ yields an r that changes only very slowly (where the trend is clear only with a very long sample).

Similarly:

$$\begin{aligned}\dot{N}_t^M &= 0\\ \dot{N}_t^S &= \frac{g\bar{S}}{B_S F(k,1) X_t}\end{aligned}$$

OK, but now what happens if $\bar{A}B_S \neq \bar{S}B_A$? What are the transitional dynamics? Write the Hamiltonian:

$$\begin{aligned}H(t) &= e^{-\rho t}\frac{\left[\left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta}\right]^{1-\sigma}-1}{1-\sigma}\\ &\quad +\lambda\left[B_M F(K(t),X)-\delta K(t)-M(t)-P_A A(t)-P_S S(t)\right]\end{aligned}$$

Take first order conditions

$$\begin{aligned}\frac{\partial H}{\partial M} &= \frac{\gamma}{M}e^{-\rho t}\left[\left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta}\right]^{1-\sigma}-\lambda=0\\ \frac{1}{P_S}\frac{\partial H}{\partial S} &= \frac{\theta}{S+\bar{S}}e^{-\rho t}\left[\left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta}\right]^{1-\sigma}-\lambda=0\\ \frac{1}{P_A}\frac{\partial H}{\partial A} &= \frac{\beta}{A-\bar{A}}e^{-\rho t}\left[\left(A(t)-\bar{A}\right)^{\beta} M(t)^{\gamma}\left(S(t)+\bar{S}\right)^{\theta}\right]^{1-\sigma}-\lambda=0\\ \frac{\partial H}{\partial K} &= B_M F_K(k,1)-\delta\end{aligned}$$

Thus:

$$\begin{aligned}P_S S(t) &= \frac{\theta}{\gamma}M(t)-\frac{B_M}{B_S}\bar{S}\\ P_A A(t) &= \frac{\beta}{\gamma}M(t)+\frac{B_M}{B_A}\bar{A}\end{aligned}$$

Let

$$\begin{aligned}\kappa &\equiv \left(\frac{B_A}{B_M}\frac{\beta}{\gamma}\right)^{\beta}\left(\frac{B_S}{B_M}\frac{\theta}{\gamma}\right)^{\theta}\\ \varepsilon &\equiv \frac{B_M}{B_S}\bar{S}-\frac{B_M}{B_A}\bar{A}\end{aligned}$$

Then we can re-write the Hamiltonian as follows:

$$H(t) = e^{-\rho t}\frac{[\kappa M]^{1-\sigma}-1}{1-\sigma} + \lambda\left[B_M F(K(t),X) - \delta K(t) - \frac{1}{\gamma}M(t) - \varepsilon\right]$$

And, again, take FOC (replacing F with a Cobb-Douglas function):

$$\begin{aligned}
e^{-\rho t}\kappa^{1-\sigma}M^{-\sigma} &= \frac{\lambda}{\gamma} \Rightarrow M = \left[\frac{\lambda}{\gamma}e^{\rho t}\kappa^{\sigma-1}\right]^{-\frac{1}{\sigma}} \\
-\frac{\dot{\lambda}}{\lambda} &= B_M \alpha K^{\alpha-1}X^{1-\alpha} - \delta \\
\dot{K} &= B_M K^{\alpha}X^{1-\alpha} - \delta K(t) - \frac{1}{\gamma}M(t) - \varepsilon \\
&= B_M K^{\alpha}X^{1-\alpha} - \delta K(t) - \frac{1}{\gamma}\left[\frac{\lambda}{\gamma}e^{\rho t}\right]^{-\frac{1}{\sigma}}\kappa^{\frac{1-\sigma}{\sigma}} - \varepsilon
\end{aligned}$$

Define:

$$\tilde{\lambda}X^{-\sigma}e^{-\rho t}\kappa^{1-\sigma} = \lambda;\ \tilde{K} = \frac{K}{X}$$

Then:

$$\begin{aligned}
\frac{\dot{\tilde{\lambda}}}{\tilde{\lambda}} &= -\alpha B_M \tilde{K}^{\alpha-1} + \delta + \rho + \sigma g \\
\frac{\dot{K}}{K} &= B_M \frac{X}{K}^{-\alpha} - \delta(t) - \frac{1}{\gamma X}M(t) - \frac{\varepsilon}{X} \\
&= B_M \frac{X}{K}^{-\alpha} - \delta - \frac{1}{\gamma X}\left[\frac{\lambda}{\gamma}e^{\rho t}\right]^{-\frac{1}{\sigma}}\kappa^{\frac{1-\sigma}{\sigma}} - \frac{\varepsilon}{X} \\
\frac{\dot{\tilde{K}}}{\tilde{K}} &= B_M \tilde{K}^{\alpha-1} - (\delta + g) - \frac{1}{\gamma}\left[\frac{\tilde{\lambda}}{\gamma}\right]^{-\frac{1}{\sigma}}\tilde{K}^{-1} - \frac{\varepsilon}{X}
\end{aligned}$$

If there is time: Talk through Figures 4-6 (reproduced here). The figure plots out the $\tilde{K}$ and $\tilde{\lambda}$ loci. The X term is what is causing the $\tilde{K}$ loci to shift over time. Assuming ε is less than 0, as time passes the X term gets bigger, meaning $-\frac{\varepsilon}{X}$ gets smaller; this will shift down the $\tilde{K}$ loci as time passes.

Now consider the phase diagram for a given $\tilde{\lambda}$, $\tilde{K}$ pair (i.e., imagine that we could hold $\frac{\varepsilon}{X}$ fixed for a moment). What would the transitional dynamics look like for the phase diagram? Looking at the first equation, if $\tilde{K} > K^*$ then $\tilde{\lambda}$ is positive (negative otherwise). Looking at the second equation, if we are above the $\tilde{K}_\tau$ locus, then $\tilde{K}$ is positive (negative otherwise). Thus we can draw the transitional dynamics as in Figure 5.

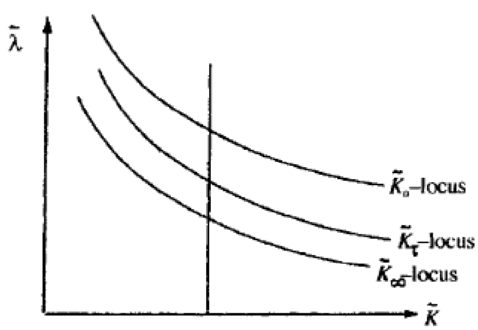


FIGURE 4
Time varying loci

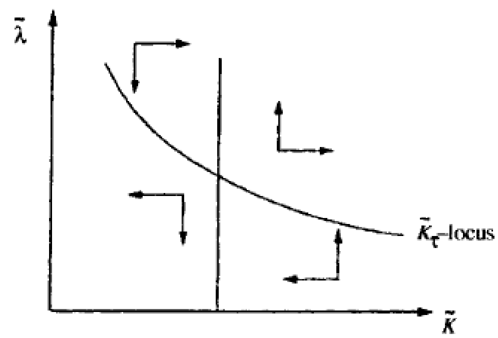
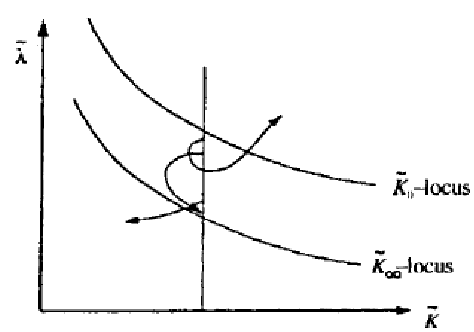


FIGURE 5
Directions for immediate movements at time τ



But now there is the additional complication that the $\tilde{K}$ locus is moving around? What is the economic intuition for this loop? (With $\varepsilon = 0$, we would have the interest rate monotonically converging from below to the steady-state interest rate). The real interest rate increases first and then comes back to the original level.

Notes on Ngai and Pissarides (2007):

Here, the empirical facts that we would like to fit are similar to the Kuznets facts: movement out of certain industries (e.g., agriculture), and towards others (e.g., services). There is an additional fact that Ngai and Pissarides mention: that the changes in the relative consumption of different types of goods is much more stark in nominal rather than real (where we take into account the movements in relative prices of different goods) terms. In other words, over time $\frac{P_s C_s}{P_a C_a}$ changes a lot more than $\frac{C_s}{C_a}$.

Overview of the model: one industry produces capital goods and consumption goods; all others only produce different consumption goods. Industries have identical production functions, except for differential rates of productivity growth. Preferences are nested CES. There is a constant intertemporal elasticity of substitution and a different constant elasticity of substitution across products within periods.

The definition of a balanced growth path here is somewhat different than in Kongsamut et al. It is not a requirement that the real interest rate is constant. Rather we seek conditions under which the aggregate ratios (e.g., $\frac{k}{y}$, $\frac{c}{y}$ are constant). In fact, for Ngai and Pissarides the real interest rate will not be constant. The second part of the goal is to have differential growth rates of employment across industries, depending on industries' TFP growth rates.

Preferences are similar to Kongsamut et al, but without the non-homotheticity terms.

$$\begin{aligned} U &= \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \text{ where} \\ C(t) &= \left[\sum_{i=1}^m \omega_i \left(c_i\right)^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon - 1}} \end{aligned}$$

Here ε is the elasticity of substitution across products; θ is the inverse intertemporal elasticity of substitution.

On the production side of the economy, there are m sectors. In these equations n_i is the fraction of the labor force (which grows at rate υ[3]) that is employed in industry i; the capital-labor ratio in industry i is k_i.

Each industry has nearly identical production functions:

[3] In class, we had set $\upsilon = 0$.

$$c_i = A_i F(n_i k_i, n_i) = A_i k_i^{\alpha} n_i,$$

for all industries except for industry m. For industry m:

$$c_m + \dot{k} + (\delta + \upsilon)\, k = A_m k_m^{\alpha} n_m$$

Define γ_i as the TFP growth rate in industry i. The final building block of the model are the labor market clearing conditions. The labor-market clearing condition and capital-market clearing conditions state that

$$\sum n_i = 1 \;\; ; \;\; \sum n_i k_i = k$$

This is it for the set-up of the model.

We will solve for the equilibrium allocation in a few steps. We will write out the first-order conditions of the planner's problem, and then use these expressions to write out the evolution of consumption, employment, and output in each industry. We will then use these expressions to find conditions under which employment shares are changing over time (i.e., when there is structural change), and conditions under which the aggregate economy has a balanced growth path (where consumption and capital per worker grow at constant rates.) Finally, we will write out the long-run implications (for the labor, consumption, and output shares) of the balanced growth path).

Step 1:

Define p_i as the price of good i. Since all goods have identical production functions, with identical factor intensities, the relative prices for any two goods will just be inversely related to their TFP terms. In particular,

$$\frac{p_i}{p_m} = \frac{A_m}{A_i}$$

Consider, first the first-order conditions of the consumer that is choosing among the m products: The demand curves that follow from this maximization problem are

$$\frac{p_i c_i}{p_m c_m} = \left(\frac{p_i}{p_m}\right)^{1-\varepsilon} \left(\frac{\omega_i}{\omega_m}\right)^{\varepsilon} = \left(\frac{A_m}{A_i}\right)^{1-\varepsilon} \left(\frac{\omega_i}{\omega_m}\right)^{\varepsilon} = x_i$$

Here, x_i is the consumption expenditure share of good i relative to that of the manufactured good. For future reference define Define $X \equiv \sum x_i$ and c and y as aggregate consumption and output (again relative to manufacturing):[4]

$$
\begin{aligned}
c &\equiv \sum_{i=1}^{m} \frac{p_i c_i}{p_m} \\
y &\equiv \sum_{i=1}^{m} \frac{p_i F^i}{p_m} \\
&= \sum_{i=1}^{m} \frac{A_m F^i}{A_i} = \sum_{i=1}^{m} \frac{A_m}{A_i} A_i k_i^{\alpha} h_i \\
&= \sum_{i=1}^{m} \frac{A_m}{A_i} A_i k_m^{\alpha} h_i = A_m k^{\alpha}
\end{aligned}
$$

The dynamic efficiency condition states that:[5]

$$
\begin{aligned}
\frac{\dot{u}_{c_m}}{u_{c_m}} &= -\alpha A_m k_m^{\alpha-1} + (\delta + \rho + \upsilon) \\
-\theta \frac{\dot{c}}{c} + \left(\frac{1-\theta}{\varepsilon - 1} \right) \frac{\dot{X}}{X} &= -\alpha A_m k_m^{\alpha-1} + (\delta + \rho + \upsilon)
\end{aligned}
$$

Where does this equation come from? Write out the Hamiltonian, as in the notes of Kongsamut et al. Take first order conditions; this expression comes from the $[c_m]$ FOC, and plugging in $\frac{\dot{\lambda}}{\lambda}$ at the appropriate point.

Step 2: Expressions for consumption, employment, output, across industries

[4] Use the equations at the beginning of "Step 2" to get from the third-to-last equation to the penultimate equation.

[5] Side note: Take first-order conditions with respect to c_m

$$
\begin{aligned}
u_{c_m} &= (\omega_m)^{\frac{\varepsilon}{\varepsilon-1} 1-\theta} \left(\frac{c}{c_m} \right)^{\frac{1-\theta}{\varepsilon-1}} c^{-\theta} \\
&= (\omega_m)^{\frac{\varepsilon}{\varepsilon-1} 1-\theta} X^{\frac{1-\theta}{\varepsilon-1}} c^{-\theta} \\
\log(u_{c_m}) &= \left(\frac{\varepsilon}{\varepsilon-1} \right) (1-\theta) \log(\omega_m) - \frac{1-\theta}{\varepsilon-1} \log X - \theta \log c
\end{aligned}
$$

Differentiate with respect to time:

$$
\frac{\dot{u}_{c_m}}{u_{c_m}} = -\theta \frac{\dot{c}}{c} + \left(\frac{1-\theta}{\varepsilon-1} \right) \frac{\dot{X}}{X}
$$

From the k FOCs:

$$\begin{aligned}\frac{p_i}{p_m} &= \frac{F_K^m}{F_K^i} = \frac{A_m}{A_i}\left(\frac{k^m}{k^i}\right)^{\alpha-1} \\ &= \frac{F_N^m}{F_N^i} = \frac{A_m}{A_i}\left(\frac{k^m}{k^i}\right)^{\alpha}\end{aligned}$$

These equations imply $k^m = k^i = k$ for all industries.

What about the n's? For $i \neq m$, $c_i = A_i k_i^a n_i = A_i k^a n_i$. Thus:

$$n_i = \frac{k^{-\alpha}}{A_i} c_i = \frac{k^{-\alpha}}{A_m}\frac{c_i p_i}{p_m} = \frac{1}{y}\frac{c_i p_i}{p_m} = \frac{c_m}{y}\frac{c_i p_i}{c_m p_m} = \frac{c_m}{y} x_i = \frac{c}{y}\frac{x_i}{X}$$

The n's need to sum to 1 and $\sum \frac{x_i}{X} = 1$, thus:

$$n_m = \frac{c}{y}\frac{x_m}{X} + \left(1 - \frac{c}{y}\right)$$

From the definition of x_i:

$$\frac{\dot{x}_i}{x_i} = (1-\varepsilon)(\gamma_m - \gamma_i) \text{ and } \frac{\dot{X}}{X} = (1-\varepsilon)(\gamma_m - \bar{\gamma})$$

Thus:

$$\frac{\dot{n}_i}{n_i} = \frac{\dot{\overline{c/y}}}{c/y} + (1-\varepsilon)(\bar{\gamma} - \gamma_i)$$

Since $\sum n_i = 1$ (and thus $\dot{n}_m = -\sum_{i \neq m} n_i$), we can solve for $\dot{n}_m$

$$\dot{n}_m = (1-\varepsilon)(\bar{\gamma} - \gamma_m)\frac{c}{y}\frac{x_m}{X} - \dot{\overline{c/y}}\left(1 - \frac{x_m}{X}\right)$$

Step 3: Conditions for structural change

There are two ways in which we can have employment shares changing over time. Either $\frac{c}{y}$ is changing over time (so that employment in manufacturing is either shrinking or growing over time), or $\frac{c}{y}$ is constant but the employment shares are changing around for the consumption sectors. From before, we know that

$$\frac{\dot{n}_i}{n_i} - \frac{\dot{n}_j}{n_j} = (1-\varepsilon)\left(\frac{\dot{p}_i}{p_i} - \frac{\dot{p}_j}{p_j}\right) = (1-\varepsilon)(\gamma_j - \gamma_i)$$

So structural change requires either that $\frac{c}{y}$ is changing or that both $\varepsilon \neq 1$ and $\gamma_i \neq \gamma_j$ for at least one pair of consumption industries.

Compare the above equation to:

$$\frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j} = \varepsilon\left(\gamma_i - \gamma_j\right)$$

Ngai and Pissarides ague that $\frac{\dot{c}_i}{c_i} - \frac{\dot{c}_j}{c_j}$ changes a lot less than $\frac{\dot{n}_i}{n_i} - \frac{\dot{n}_j}{n_j}$. This indicates that $\varepsilon \ll 1$.

Step 4: Conditions for an aggregate balanced growth path

We know that:

$$\begin{aligned}
\dot{k} &= A_m k_m^{\alpha} n_m - c_m - (\delta + \upsilon) k \\
\frac{\dot{k}}{k} &= A_m k^{\alpha - 1} n_m - \frac{c_m}{k} - (\delta + \upsilon)
\end{aligned}$$

Substitute ($n_m = 1 - \sum n_i$, and $c_m = c / \sum x_i$)

$$\begin{aligned}
\frac{\dot{k}}{k} &= A_m k^{\alpha-1}\left(\frac{c}{y}\frac{x_m}{X} + \left(1 - \frac{c}{y}\right)\right) - \frac{c_m}{k} - (\delta + \upsilon) \\
&= A_m k^{\alpha-1} - \frac{c}{y} A_m k^{\alpha-1}\left(1 - \frac{x_m}{X}\right) - \frac{c}{k}\frac{1}{X} - (\delta + \upsilon) \\
&= A_m k^{\alpha-1} - \frac{c}{k}\left(1 - \frac{1}{X}\right) - \frac{c}{k}\frac{1}{X} - (\delta + \upsilon) \\
&= A_m k^{\alpha-1} - \frac{c}{k} - (\delta + \upsilon)
\end{aligned}$$

Also from before, we have that:

$$\begin{aligned}
-\theta\frac{\dot{c}}{c} + \left(\frac{1-\theta}{\varepsilon - 1}\right)\frac{\dot{X}}{X} &= -\alpha A_m k_m^{\alpha-1} + (\delta + \rho + \upsilon) \\
-\theta\frac{\dot{c}}{c} - (1-\theta)(\gamma_m - \bar{\gamma}) &= -\alpha A_m k_m^{\alpha-1} + (\delta + \rho + \upsilon) \\
\theta\frac{\dot{c}}{c} &= \alpha A_m k_m^{\alpha-1} - (\delta + \rho + \upsilon) + (\theta - 1)(\gamma_m - \bar{\gamma})
\end{aligned}$$

Thus, we need $\theta = 1$ for c and k to be growing at the same (constant) rate. So, the definition of a BGP is different from that in Kongsamut et al (2001) (which relied on a constant interest rate). With a unitary intertemporal elasticity of substitution, consumption is a constant fraction of the present value of wealth (it is independent of the interest rate).[6]

[6] Side Calculations. Consider a consumer who faces a constant real interest rate R, and define the present value of consumption as W.

If the utility function is:

$$u(c_0, c_1, c_2, c_3 \ldots.) = \sum_{t=0}^{\infty} \beta^t \frac{(c_t)^{\frac{\sigma-1}{\sigma}}}{\frac{\sigma-1}{\sigma}}$$

Step 5: Long-run evolution of employment, consumption, output shares.

Now, finally, we are interested in the long-run evolution of employment and output. It will be useful to describe how average productivity, $\bar{\gamma}$ evolves over time. Remember that the definition of $\bar{\gamma} = \sum \frac{x_i}{X}\gamma_i$. What we are interested in is how this average productivity growth changes over time. Taking derivatives of this expression, with respect to time, we have that:

$$\frac{d\bar{\gamma}}{dt} = \sum \frac{x_i}{X}\gamma_i \left(\frac{\dot{x}_i}{x_i} - \sum \frac{\dot{x}_j}{X} \right) = ... = -(1-\varepsilon) \sum \frac{x_i}{X} (\gamma_i - \bar{\gamma})^2$$

The first equality is an application of the quotient rule. The second equality applies $\frac{\dot{x}_i}{x_i} = (1-\varepsilon)(\gamma_i - \gamma_m)$ and then some substitutions. What is important about this expression is that average productivity in the economy is monotonically decreasing (increasing) provided ε is less than (greater than) 1. When ε is smaller than 1, more and more of the economy's resources will be devoted to industries that have low productivity growth, pulling down the average over time. The only way this can be true, for all time periods, is for $\bar{\gamma}$ to approach $\min \gamma_i$ from above.

Notice that employment in all but one of the consumption industries will tend to 0 in the long run. Why is this? Remember that:

$$\frac{\dot{n}_i}{n_i} = (\varepsilon - 1)(\gamma_i - \bar{\gamma}) + \frac{\dot{\overline{c/y}}}{c/y}$$

The second term will go to 0 on a BGP (because c, y, and k all grow at the same rate) . What about the first term? Suppose we are in the $\varepsilon < 1$ case. Then this term will be negative for all industries with faster than average productivity growth, and negative for all industries with slower than average productivity growth. We also know that $\bar{\gamma}$ is decreasing over time, heading in the limit to the $\min \gamma_i$, so that more and more industries have higher than average productivity growth, as time goes by. Hump-shaped employment is possible (was not in the Kongsamut et al. paper), because γ_i could at first be smaller than $\bar{\gamma}$ and

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The first order conditions are:

$$c_{t+1} = \beta^{\sigma}(1+R)^{\sigma} c_t$$

Manipulating the intertemporal budget constant:

$$\begin{aligned} W &= c_0 \sum_{k=0}^{\infty} \left(\beta^{-\sigma} (1+R)^{1-\sigma} \right)^k \\ &= \frac{c_0}{1 - \beta^{-\sigma}(1+R)^{1-\sigma}} \\ c_0 &= W \cdot \left[1 - \beta^{-\sigma} (1+R)^{1-\sigma} \right] \end{aligned}$$

Thus, consumption is independent of R if $\sigma = 1$.

then bigger than $\bar{\gamma}$. Thus $\frac{\dot{n}_i}{n_i}$ is negative for more and more industries, and is positive only for the industry with the minimum γ. In other words, $n_i^* \equiv \lim n_i = 0$ for all consumption industries that have productivity growth above the minimum.

So the only thing to work out is the employment share of the least-productive consumption industry, and the employment share in manufacturing.

Take the equations that we worked out before. Then define consumption and the capital stock in terms of efficiency units: $c_e \equiv c A_m^{-1/(1-\alpha)}$ and $k_e \equiv k A_m^{-1/(1-\alpha)}$:

$$\begin{aligned} \frac{\dot{c}_e}{c_e} &= \left[a k_e^{\alpha-1} - (\delta + \upsilon + \rho)\right] - \frac{\gamma_m}{1-\alpha} \\ \dot{k}_e &= k_e^\alpha - c_e - \left(\frac{\gamma_m}{1-\alpha} + \delta + \upsilon\right) k_e \end{aligned}$$

Manipulating these equations (looking for the BGP where $\dot{c}_e = \dot{k}_e = 0$ and using a $*$ to denote the BGP values):

$$\begin{aligned} (k_e^*)^{\alpha-1} &= \frac{1}{\alpha}\left[\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho\right] \\ c_e^* &= (k_e^*)^\alpha - \left(\frac{\gamma_m}{1-\alpha} + \delta + \upsilon\right) k_e^* \end{aligned}$$

Solving this system equations gives up c_e^* and k_e^*. From here, we know how much consumption is occurring along the BGP, and how much labor must be allocated to produce this consumption good. Proposition 5:

$$\begin{aligned} n_c^* &= 1 - \frac{\alpha\left(\frac{\gamma_m}{1-\alpha} + \delta + \upsilon\right)}{\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho} \\ n_m^* &= \frac{\alpha\left(\frac{\gamma_m}{1-\alpha} + \delta + \upsilon\right)}{\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho} \end{aligned}$$

Side Calculations:

$$\begin{aligned} k_e^* &= \left[\frac{\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho}{\alpha}\right]^{\frac{1}{\alpha-1}} \\ c_e^* &= \left[\frac{\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} \underbrace{\left[1 - \frac{\alpha\left(\frac{\gamma_m}{1-\alpha} + \delta + \upsilon\right)}{\frac{\gamma_m}{1-\alpha} + \delta + \upsilon + \rho}\right]}_{n_c^*} \end{aligned}$$

In the $\varepsilon < 1$ case, even though n_i is equal to 0 for all but two sectors, consumption (and output) are still positive (and bounded away from 0) for every sector. The faster productivity growth outweighs the decline in employment shares for the faster-growing consumption industries. In particular:

$$\begin{aligned}
\frac{\dot{F}^i}{F^i} &= \gamma_i + \alpha \frac{\dot{k}_i}{k_i} + \frac{\dot{n}_i}{n_i} \\
&= \gamma_i + \frac{\gamma_m \alpha}{1-\alpha} + (\varepsilon - 1)(\gamma_i - \bar{\gamma}) \\
&= \varepsilon \gamma_i + \frac{\gamma_m \alpha}{1-\alpha} + (1-\varepsilon)\bar{\gamma},
\end{aligned}$$

which is positive for all sectors provided $\varepsilon < 1$.

When $\varepsilon > 1$, $\bar{\gamma}$ is increasing over time, and some of the slower-growth consumption sectors will disappear (If α is small enough, and ε is large enough, $(1-\varepsilon)\bar{\gamma} + \varepsilon\gamma_i$ will be negative).