Lecture Notes 2: Exchangeability

i. definitions

To motivate these notes, consider the standard linear model

\[ y_i = X_i \beta + \varepsilon_i \]  

(2.1)

It is common to assume that the errors are normal and independent and identically distributed (n.i.d.) or that the errors are i.i.d. Assumptions of this type do not naturally correspond to the substantive social science one brings to an empirical exercise. Similar comments can be made about more general data generating processes than (2.1) of course.

I want to argue that the substantive social science knowledge one brings to an analysis is more comfortably associated with the notion of exchangeability. The definition of exchangeability employs the permutation operator \( \rho(i) \) which rearranges any set of integers.

**Definition 2.1. Exchangeability**

A collection of random variables \( \varepsilon_i \) is exchangeable if for every finite subset of the random variables, \( \varepsilon_i, \ldots, \varepsilon_{i_N} \) and every permutation operator \( \rho(\cdot) \),

\[ \mu\left( \varepsilon_i \leq a_1, \ldots, \varepsilon_{i_N} \leq a_N \right) = \mu\left( \varepsilon_{\rho(i)} \leq a_1, \ldots, \varepsilon_{\rho(i_N)} \leq a_N \right) \]  

(2.2)
Exchangeability thus means that there are symmetries in the joint probabilities associated with elements of the sequence, i.e. the subscripts which identify particular observations do not matter. Clearly, if the collection \( \varepsilon_i \) is i.i.d., then it is exchangeable. Exchangeability, however, does not imply independence. This is easily seen for the case

\[
\varepsilon_i = \eta_i + \xi
\]  

(2.3)

where \( \eta_i \) is i.i.d. across \( i \) and \( \xi \) is a single random variable that is independent of \( \eta_i \forall i \). DeFinetti’s theorem, discussed below, will provide a deeper link between exchangeability and independence.

The exchangeability definition may be generalized in a number of ways, typically to produce what is called conditional or partial exchangeability.

**Definition 2.2. Conditional exchangeability**

Suppose that each \( \varepsilon_i \) is associated with a vector \( X_i \). These elements of \( X_i \) may be realizations of a stochastic process. The collection of random variables \( \varepsilon_i \) is conditionally exchangeable if for any finite collection \( \varepsilon_{i_1}...\varepsilon_{i_N} \) and every permutation operator \( \rho(\cdot) \),

\[
\mu\left(\varepsilon_{i_1} \leq a_{i_1},...,\varepsilon_{i_N} \leq a_{i_N} \mid X_{i_1} = X_{i_2} = \ldots = X_i\right) = \\
\mu\left(\varepsilon_{\rho(i_1)} \leq a_{\rho(i_1)},...,\varepsilon_{\rho(i_N)} \leq a_{\rho(i_N)} \mid X_{i_1} = X_{i_2} = \ldots = X_i\right)
\]  

(2.4)

One can easily modify the notion of conditional exchangeability, for example to requiring that \( X_i \) are not equal but rather lie in a common set \( \bar{X} \), conditional exchangeability with respect to this set holds if for every permutation operator \( \rho(\cdot) \)
This latter definition, in turn, is suggestive of an important use of conditional exchangeability, which is to partition observations into exchangeable classes. For example, suppose that \( X_i \) encodes an individual’s ethnicity and gender. Then conditional exchangeability of errors within ethnicity/gender combinations would lead naturally to partitioning the data along these lines.

\[ \mu(\varepsilon_1 = a_1, \ldots, \varepsilon_N = a_N | X_1 \in \bar{X}, X_2 \in \bar{X}, \ldots, X_i \in \bar{X}) = \]
\[ \mu(\varepsilon_{r(1)} = a_1, \ldots, \varepsilon_{r(N)} = a_N | X_1 \in \bar{X}, X_2 \in \bar{X}, \ldots, X_i \in \bar{X}) \]

(2.5)

**ii. Why exchangeability?**

I emphasize exchangeability as an idea because it pushes a researcher to think about model specification in terms of the implied residuals in a model. For example, omitted variables are easily interpreted as an exchangeability violation, conditional on the variable, the errors in the misspecified regression are no longer exchangeable. Similarly, parameter heterogeneity also leads to nonexchangeability. For these cases, the failure of nonexchangeability calls into question the interpretation of the regression. Brock and Durlauf argue that exchangeability produces a link between substantive social science knowledge and error structure, i.e. this knowledge may be used to evaluate the plausibility of exchangeability. They suggest that a good empirical practice would for researchers to question whether the errors in a model are exchangeable, and if not, determine whether the violation invalidates the purposes for which the regression is being used. This cannot be done in an algorithmic fashion, but as is the case with empirical work quite generally, requires judgments by the analyst. Draper et al (1993) make an argument on the primacy of exchangeability in statistical model building; see also McCullagh (2005).

To be clear, exchangeability is not necessary for a statistical model to be interpretable as a behavioral structure. Heteroskedasticity of regression errors violates exchangeability, but does not affect the interpretability of regression coefficients per se. Rather, exchangeability represents a criterion by which a researcher can evaluate his
modeling choices. If a researcher believes that the errors in his model are not exchangeable, then good empirical practice requires that he consider whether the violations invalidate the substantive claims for which the model will be used. This inevitably leads to issues of judgment, but judgments are inevitable in any substantive empirical exercise. In turn an emphasis on judgment helps explain why exchangeability notions are more common in Bayesian than frequentist contexts. For our purposes, what matters is that exchangeability provides a benchmark for assessing model specification, since \( \varepsilon_i \) is defined relative to a model.

### iii. DeFinetti’s theorem

Remarkably, exchangeability also provides a justification for i.i.d. assumptions. This property is known as DeFinetti’s theorem as the first formulation of the result (for binary random variables) is due to Bruno DiFinetti.

**Theorem 2.1. DeFinetti’s Theorem**

Suppose that \( \varepsilon_i \) is an infinite real-valued exchangeable sequence. Then there exists a probability measure \( Q \) on the space of scalar random variable probability measures \( F \) such that

\[
\mu(\varepsilon) = \int_F \prod_i F(\varepsilon_i) dQ(F)
\]  

(2.6)

In words, DeFinetti’s theorem states that the probability measure describing any infinite exchangeable sequence can be written as mixture of i.i.d. probability measures. Each sample path realization will obey one of the probability measures, so each sample path will behave as an i.i.d. sequence.

This formulation of DeFinetti’s Theorem is taken from Bernard and Smith (1994), who provide a heuristic description of its proof. A full proof of along with many related results may be found in Kallenberg (1997). DeFinetti’s Theorem does not apply to finite
exchangeable sequences; the standard example is a sequence of draws of balls from an urn consisting of a finite number of red and half black balls; the probabilities of a sequence of colors is exchangeable, but the probability of a given color for the next draw will depend on what colors have already been removed.
References


