Lecture Notes 10

Understanding the Determinants of Interest Rates and Stock Prices

So far, we have discussed risk free assets, i.e. there is no uncertainty in their returns. However, the returns on virtually all assets are risky.

How does one think of risk? Suppose we have an assets $a$ and $b$ with associated returns $r_{a,t}$ and $r_{b,t}$ at time $t$.

Think of these returns as representing the actual rates of return between $t$ and $t+1$. In general, we will not know what these are with certainty at time $t$. 
Holding Returns

In discussing returns on assets between $t$ and $t+1$, I am referring to the one period *holding returns* on the asset. The idea is that I invest a dollar in an asset today and either the asset matures (for example, a 1 period government bond comes due) or I resell the asset.
Why are these holding returns risky? Here are two examples.

For a government bond, the nominal interest rate on a 1 period bond is known at time $t$; it is $i_t$. However, the real return is

$$r_t = i_t - \pi_{t+1}$$

where $\pi_t$ is the inflation rate between $t$ and $t+1$. The inflation rate is not known with certainty at time $t$, so the real return is risky.
For stocks, suppose I buy one share of a stock $a$ at price $P_{a,t}$. Suppose that at time $t+1$ I receive a dividend $D_{a,t+1}$ and sell the share. My one period holding return is

$$\frac{P_{a,t+1} + D_{a,t+1} - P_{a,t}}{P_{a,t}}$$

At time $t$, I will not know the values of $P_{a,t+1}$ and $D_{a,t+1}$ with certainty. Hence the *nominal* holding return is risky. The real return will also be affected by inflation.

Let’s now return to the question of how to measure risk, focusing on holding returns.
To understand risk, it is necessary to distinguish between the expected holding return on an asset and its actual return.

By expected, I refer to a particular mathematical property, but we need not worry about what it means at this point.

The expected return is related to what one expects to happen on average.
Here is an example. Suppose that I will pay you $1 if a coin flip lands heads whereas you will pay me $1 if the coin lands tails.

Each time we flip the coin, you will either win or lost $1.

But on average, you will win $0.
Using this idea of expected value, we can think about the return on an asset $a$ as consisting of two parts: the expected return and a residual that reflects the fact that it is risky.

In other words

$$ r_{a,t} = E(r_{a,t}) + \varepsilon_{a,t} $$

where

$E(r_{a,t})$ is the expected return on the asset and

$\varepsilon_{a,t}$ is the “excess” return.

Comment: $E(\varepsilon_{a,t})$ always equals 0.
To say that asset $a$ is risk free is to say that $\varepsilon_{a,t}$ always equals zero.

The riskiness of an asset will depend upon the properties of $\varepsilon_{a,t}$.

Further, we would intuitively expect that riskier assets are associated with higher expected returns? This is so when individuals are risk averse.

To understand how risk relates to higher expected returns, we need to ask why an individual is willing to hold a riskier asset when less risky assets are available. How is an individual compensated for holding a riskier asset? By higher expected returns!
But how does one compare the riskiness of different assets? Riskiness will depend on the properties of $\varepsilon_{a,t}$.

One way to evaluate riskiness is in terms of the variance of $r_{a,t}$ (denoted as $\text{Var}(r_{a,t})$). Notice that the variance is computed conditional on information at $t$. The term variance has a precise mathematical definition, but for our purposes it is sufficient to understand it as measuring how much $\varepsilon_{a,t}$ moves around 0 (its expected value.)

Example. Suppose that $r_{a,t} = 100$ with probability $1/2$ and $r_{a,t} = -100$ with probability $1/2$. Suppose we compare it to an $r_{b,t}$ which has the property that $r_{b,t} = 10$ with probability $1/2$ and $r_{b,t} = -10$ with probability $1/2$. 
$r_{a,t}$ has a higher variance than $r_{b,t}$, even though each has an expected value of 0.

Does $\text{Var}_t(r_{a,t+1}) > \text{Var}_t(r_{b,t+1})$ imply that asset $a$ is riskier than asset $b$? Theoretically, the answer is no.

Recall the following question from Problem Set 1

3. Suppose you are worker at Ford motor company. Assume that stock in the Ford Motor company has an expected return of 5%; however, with probability .5 the return is 4% and with probability .5 the return is 6%. In contrast, Honda stock has an expected return of 5%; with probability .5 the return is 3.8% and with probability .5 and 6.2% with probability .5. Which stock would you buy? Would you answer differ if you were a college professor? Relate your answer to the issue of risk versus return in personal investments.

The purpose of the question was to indicate that risk cannot be properly measured independently of the other factors.
Key Idea in Understanding Risk

To an individual, the riskiness of one asset is determined its covariance of the return on the asset with the rest of his portfolio. So, think about your portfolio as having two parts, a given asset and everything else. You can think of “everything else” as an asset itself.

Note, covariance between any two returns is denoted $\text{Cov}(r_{a,t}, r_{b,t})$. Intuitively, covariance describes how one random object moves with another. When two random objects are independent, the covariance is 0.
The covariance of assets $a$ and $b$ measures whether the returns on the assets move together or not. When the covariance is positive, then $r_{a,t}$ tends to be high when $r_{b,t}$ is high, etc.

So, for the Ford worker, Honda stock may be less risky than Ford stock because Honda stock diversifies his portfolio, i.e. off at higher rates when the rest of his portfolio is doing badly. In portfolio, I am including wages!

Notice that a college professor will not assess risk in the same way.

Let’s be a bit more formal
Suppose that you currently hold an asset, call it $a$, that pays the following holding returns

10% with probability .5 and 0% with probability .5

Suppose you want to compare two other assets, $b$ and $c$

Asset $b$ pays holding returns according to

7.5% with probability .5, 2.5% with probability .5

The returns are independent of $a$ and $c$

Asset $c$ pays

10% with probability .5 and 0% with probability .5
However, asset $c$ always pays 10% when asset $a$ pays 0; this implies that asset $c$ always pays 0% when asset $a$ pays 10%.

Notice that a portfolio that consists of $\frac{1}{2}$ asset $a$ and $\frac{1}{2}$ asset $c$ will pay 5% with probability 1.

Bottom line: asset $c$ is less risky than asset $b$ from the perspective the individual we started with.

Formally, the holding returns on assets $a$ and $c$ negatively covary, so holding them together reduces aggregate risk.
Capital Asset Pricing Model

There is a famous formula that relates the risk of assets to their expected returns in a financial market equilibrium, known as the capital asset pricing model.

This formula does not fully (empirically) explain the return structure to different assets, but does have explanatory power. It is often used in internal investment calculations.

The formula is based on considering a financial market consisting of many different assets.
Let

\[ E(r_{a,t}) \] denotes the expected return of any asset \( a \) given information available at \( t \)

\[ E(r_{rf,t}) \] denotes the expected return of a risk free asset \( rf \) (This may not exist, but one can infer what the expected return would be if it did).

\[ E(r_{m,t}) \] denotes the expected return on the market as a whole. Think of a market portfolio made up of all the outstanding stocks, for example.
The Capital Asset Pricing Model (CAPM) states that the equilibrium expected asset returns will have the following relationship:

\[ E(r_{a,t}) = E(r_{rf,t}) + \frac{\text{Cov}(r_{a,t}, r_{m,t})}{\text{Var}(r_{m,t})} \left( E(r_{m,t}) - E(r_{rf,t}) \right) \]

I will not derive this formula. What is essential is the term \( \frac{\text{Cov}(r_{a,t}, r_{m,t})}{\text{Var}(r_{m,t})} \). This is also known as \( \beta_{a,t} \) and measures the riskiness of asset \( a \) relative to the market at time \( t \). The CAPM formula implies that assets which covary positively with the market return are those assets that are relatively riskier!!
We have discussed the relationship between bond prices for interest rates for a one period bonds. How do we think about interest rates and longer period bonds? In this discussion, I use nominal rates.

Suppose that a bond will pay $100 2 periods from now. Suppose the bond has a price of $P_2$. The implicit 2 period interest rate, $i_{2,t}$, also called the *yield to maturity*, is defined by:

$$P_{2,t} = \frac{100}{(1+i_{2,t}) \times (1+i_{2,t})}$$

The implicit interest rate on the bond is what is produced a constant return for two periods.
For a $K$-period bond (one that pays $100$ $K$ periods in the future, the implicit interest rate is defined by

$$P_{K,t} = \frac{100}{(1 + i_{K,t})^K}$$

What is the relationship between the interest rates for bonds of different maturities?

This relationship is called the term structure of interest rates.
How might one theorize about the term structure?

One way to think about the term is to compare the alternative strategies of buying a long term bond to buying a sequence of 1 period bonds.

For a two periods, for example, one can either buy a 2 period bond or buy a 1 period bond and reinvest. Suppose that an agent is risk neutral, so he/she only cares about the expected return. Let \( E(i_{t+1}) \) denote the agent’s expectation about the one period interest rate at \( t + 1 \).

Then interest rates much have the property that all investment strategies produce the same expected value.
This means

\[(1 + i_t) \times (1 + E(i_{t+1})) = (1 + i_{2,t}) \times (1 + i_{2,t})\]

This equality implies that approximately

\[i_{2,t} = \frac{1}{2} \left( i_t + E_t(i_{t+1}) \right)\]

This holds for bonds of different maturities.

\[i_{K,t} = \frac{1}{K} \left( \sum_{j=0}^{K-1} E_t(i_{t+j}) \right)\]

Note that \(E_t(i_t) = i_t\) always holds.

This is known as the pure expectations-based theory of the term structure of interest rates.
Key idea: long term interest rates reflect beliefs about short term interest rates.

If one graphs $i_{K,t}$ against $K$, one has the yield curve. An upward sloping yield curve implies short term rates are expected to rise in the future.

Comment: one can generalize the basic term structure to include risk premia. Suppose that $\phi_{K,t}$ denotes the risk premium for $K$ period bonds, then a risk adjusted term structure equation is

$$i_{K,t} = \phi_{K,t} + \frac{1}{K} \left( \sum_{j=0}^{K-1} E_t(i_{t+j}) \right)$$
Risk Neutrality and the Random Walk Theory of Stock Prices

One often thinks of Wall Street traders as risk neutral, i.e., stocks are held in order to maximize the present discounted value of a portfolio. Does this provide a theory of stock prices?

Suppose that dollars at $t+1$ are discounted at the rate $1+i_t$. I use nominal interest rates since everything is measured in dollars. A risk neutral trader will be willing to buy a stock so long as

$$P_t < \frac{1}{1+i_t} E_t \left( P_{t+1} + D_{t+1} \right)$$
The trader will always sell the stock if

\[ P_t > \frac{1}{1+i_t} E_t \left( P_{t+1} + D_{t+1} \right) \]

These conditions imply that means that the equilibrium stock price must obey

\[ P_t = \frac{1}{1+i_t} E_t \left( P_{t+1} + D_{t+1} \right) \]
Suppose that the time between $t$ and $t+1$ is small.

This means

1. $D_t = 0$ (approximately)

2. $i_t = 0$ (approximately)

Why? Dividends are paid infrequently and the interest rate between today and tomorrow is of course about 0.
The stock price formula for small time horizons is therefore

\[ P_t = E\left( P_{t+1} \right) \]

or

\[ P_{t+1} = P_t + \varepsilon_t \]

where \( E_t \left( \varepsilon_{t+1} \right) = 0 \)

This is the random walk theory of stock prices.