1. Population growth

Suppose that labor is not longer constant, but grows at constant rate $n$

$$L_t = (1 + n)L_{t-1}$$

As before, capital accumulation follows

$$K_t = sY_{t-1} = sAF(K_{t-1}, L_{t-1})$$
If we divide both sides of this equation by $L_{t-1}$,

$$\frac{K_t}{L_{t-1}} = sA\ell f(k_{t-1})$$

Since $L_t = (1+n)L_{t-1}$, $L_{t-1} = \frac{L_t}{(1+n)}$, this means that

$$(1+n)k_t = sA\ell f(k_{t-1})$$

or

$$k_t = \frac{s}{1+n} A\ell f(k_{t-1})$$
This is an equation that describes the evolution of the capital labor ratio in the presence of population growth.

The steady state for this model must therefore fulfill

\[
k^* = \frac{S}{1+n} Af(k^*)
\]

In the presence of population growth, capital and labor both grow at rate \( n \) in the steady state.
Again, there is no steady state growth in per worker output. Capital and labor each grow at rate $n$ in the steady state. By constant returns to scale, output grows by $n$ in the steady state. Hence output per worker is constant in the steady state.

2. Technical change

How can technical change be introduced into this model?
One way is the following. Let’s assume that the labor supply is not growing. Suppose that technical change is interpretable as “labor augmenting); in other words, technical change makes workers more productive. (This was an inspired idea by Solow). This idea may be formalized by expressing the aggregate production function as

\[ Y_t = F(K_t, A_t L_t) \]
Now suppose we measure labor not in terms of number of workers, but in terms of effective labor, i.e. rather than work with $\bar{L}$, we measure labor as

$$L_{t}^{\text{eff}} = A_{t} \bar{L}.$$  

Let

$$y_{t}^{\text{eff}} = \frac{Y_{t}}{L_{t}^{\text{eff}}}$$

and

$$k_{t}^{\text{eff}} = \frac{K_{t}}{L_{t}^{\text{eff}}}.$$
As before,

\[ K_t = sF\left(K_{t-1}, A_{t-1}\bar{L}\right) \]

Dividing both sides by \( L_{t-1}^{\text{eff}} = \bar{L}(1+g)^{t-1} \). Following exactly the same logic as for the case of population growth, then the capital/effective labor ratio is determined by

\[ k_t^{\text{eff}} = \frac{S}{1+g} f\left(k_{t-1}^{\text{eff}}\right) \]
The steady state is

\[ k^{\text{eff}} = \frac{S}{1 + g} f (k^{\text{eff}}) \]

Bottom line:

Capital grows at the rate \( g \).

Effective labor grows at the rate \( g \).

Output per worker grows at the rate \( g \).

Hence output per worker grows at \( g \).
When there is both population growth and technical change, one can show:

Capital $K$ grows by $n + g$

Output $Y$ grows by $n + g$

Output per worker $y$ grows by $g$

This provides a theory of steady state growth.
3. The Solow model with increasing returns

It is also possible to produce steady state growth in the Solow model if the aggregate production function $F$ exhibits increasing returns to scale. To illustrate, suppose that labor is constant and depreciation is 100%. Under increasing returns, $f(k)$ will be shaped as illustrated in Figure 1.
Figure A

Shape of per worker production function under increasing returns.
For such a shape, there may not be any steady state capital labor ratio. Consider Figure 2. The capital labor ratio once again evolves according to

\[ k_t = sAf(k_{t-1}) \]

However, as \( k \) increases, the amount of capital per worker begins to grow faster, not slow down as occurred in the constant returns case. This allows for sustained growth from capital accumulation. This is also known as endogenous growth, as the growth is determined inside the system.
Figure 2

No steady state capital labor ratio under increasing returns
Beyond the Solow Model

Much recent work on economic growth has focused on growth determinants that are not part of the Solow framework.

Examples of such theories are

1. Geography: factors such as lack of access to oceans, weather, disease have been argued to have important long term growth effects.
2. Rule of law. Well defined property rights, mechanisms for adjudicating disputes, etc. are all important in promoting investment.

3. Political stability. Societies which experience political instability naturally suffer from lower investment, etc.

4. Inequality. It has been argued that highly unequal societies exhibit lower growth because of implications of education of workforce, potential for political conflict and redistribution.
5. Culture. A number of authors have argued that growth requires factors such as “social capital” i.e. minimum levels of trust and trustworthiness in behavior.

6. Ethnic Diversity. Some authors have argued that the poor performance of sub-Saharan Africa is due to ethnic diversity leading to ethnic conflict, which precludes public good formation, promotes confiscation of resources, etc.

All of these are plausible, empirical evidence is not decisive however; I am skeptical on some, eg. 6