Answers to Problem Set 3

1. Since $r_{1t}$ is AR (1),

$$r_{2t} = \frac{1 + \rho}{2} r_{1t}$$

where $\rho$ is the AR(1) coefficient. Under the assumptions of the problem, $\left| \frac{1 + \rho}{2} \right| < 1$.

   a. May hold; this is immediate from the formula. Note that in treating interest rates as AR(1) we are assuming the mean has been removed, i.e. negative observations are possible.

   b. Must hold, since $\text{var}(r_{2t}) = \left( \frac{1 + \rho}{2} \right)^2 \text{var}(r_{1t})$

   c. Must hold, since $\text{var}(r_{2t} - E_t(r_{2t+1})) = \text{var}\left( \frac{1 + \rho}{2} r_{1t+1} - \frac{1 + \rho}{2} E_t(r_{1t+1}) \right)$

2. Exploiting the analogy with the dividend stock price model, it is easy to see that

$$W_{t-1} = y_{t-1} + \beta E_{t-1} W_t$$

which means that
\[ W_t - W_{t-1} = (1 - \beta)W_t - y_{t-1} + \beta(W_t - E_{t-1}W_t) \]

The projection of \((1 - \beta)W_t\) onto \(H_i(y)\) is given by the Hansen-Sargent formula. The projection of \(-y_{t-1}\) onto \(H_i(y)\) is of course \(-y_{t-1}\). The projection of \(\beta(W_t - E_{t-1}W_t)\) onto \(H_i(y)\) is

\[
\beta \left( \sum_{j=0}^{\infty} \beta^j \alpha_j \right) \epsilon_i = \beta \left( \sum_{j=0}^{\infty} \beta^j \alpha_j \right) \alpha(L)^{-1} y_i \text{ where } \alpha(L) \text{ is the fundamental } \\
\text{MA lag polynomial, which I assume is invertible. The overall projection is the sum of these.}
\]

3. Construct the time series \(P_{t+1} - P_t\). Under the random walk theory, the projection of this time series onto any Hilbert space \(H_i(Z)\) must be zero for any vector \(Z\), so long as elements of the space indexed at time \(t\) and earlier are part of the information set of agents at time \(t\).

4. \(\sigma^2_\epsilon > \sigma^2_\eta\). This follows immediately from the formula that shows how to flip roots to convert nonfundamental representations into fundamental representations. To see this, recall from lecture 3 the case where one needed to flip one root to produce a fundamental representation. Using the z-transform formula from the lecture,

\[
\sigma^2_\epsilon(z) = \sigma^2_\eta \lambda^{-2} \left( 1 - \lambda^{-1} z \right) \left( 1 - \lambda^{-1} z^{-1} \right) \prod_{k=1} (1 - \lambda_k z) \prod_{k=1} (1 - \lambda_k z^{-1})
\]

Since \(|\lambda_i| > 1\), the variance of the fundamental representation, \(\sigma^2_\eta \lambda^2\) is greater than \(\sigma^2_\eta\).

5. The available data allow one to construct \(H_i(y)\) and hence recover the forecast errors \(\epsilon_i\) where \(y_i = \alpha(L) \epsilon_i\) is the fundamental representation. The question is thus equivalent to asking whether the forecast errors may be equated with technology shocks. The answer is yes, so long as \(\pi(L)\), the mapping from technology shocks to \(y_i\), is invertible.
Note that there is no reason that this must be the case; the technology mapping is determined by the structure of the economy, not the Wold theorem!