Exercise 1: Producers.

(This problem is from a past Midterm 2.)

Consider a producer that has the following technology:

\[ y = 8K^{\frac{1}{2}}L^{\frac{1}{2}} \]

- (a) What are the returns to scale represented by this production function? (Choose between CRS, IRS, or DRS; you do not have to prove it.)
  This production function exhibits decreasing returns to scale. With Cobb-Douglas form, we can check whether the exponents are in total > (CRS), < (DRS) or = (CES) to 1: \( \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow \text{DRS} \).

- (b) (Short run) Assume that \( K = 1 \) and the firm cannot change it in the short run. Derive a condition for optimal demand for labor. Explain intuitively its economic meaning (in one sentence).
  When \( K = 1 \), \( f(K, L) = 8L^{\frac{1}{2}} \). Then our profit function is \( \Pi = 8L^{\frac{1}{2}} - wL \). To find \( L^* \), maximize \( \Pi \) with \( L \):
  \[
  \begin{align*}
  \frac{\partial \Pi}{\partial L} &= 0 \\
  \frac{4}{wL^2} - w &= 0 \\
  \implies L^* &= (\frac{4w}{w})^{\frac{1}{2}} \\
  \text{This is equivalent to our secret of happiness:} \\
  \text{MPL} &= \frac{wL^*}{p} \text{ (since MPL = } 4L^{\frac{1}{2}}) \text{. Intuitively, it means a firm should hire up to the point where the workers output is equal to what he is paid.}
  \end{align*}
  \]

- (c) Suppose that labor supply is inelastic and given by \( L^s = 16h \). Find analytically and on a graph the equilibrium wage rate.
  Equating \( L^d = MPL = \frac{wL^*}{p} = 16 \), \( \frac{wL^*}{p} = 1 \)

- (d) Find the unemployment rate with the minimum (real) wage given by \( wL/p = \frac{4}{3} \) (give one number).
  At \( \frac{wL}{p} = \frac{4}{3} \), \( L^d = 9 \) and \( L^s = 16 \). \( \frac{wL}{p} = \frac{4}{3} \)
  This results in an unemployment rate of \( \frac{16 - 9}{16} = \frac{7}{16} \).

  (before unemployment was \( \frac{16 - 9}{16} = 0 \))
• (e) Suppose \( w_L = 1 \) and \( w_K = 2 \). Derive the cost function \( c(y) \), assuming that you can adjust both \( K \) and \( L \), and plot it on the graph. Relate the shape of your cost function to the returns to scale. (Hint: The constants in this last equation are not round numbers.)

**Step 1:** Use cost minimizing condition to get the optimal combination of \( K \) and \( L \): \( \text{TRS} = \frac{-w_K}{w_L} \). Since \( \text{TRS} = \frac{MP_L}{MP_K} = \frac{2k^2yL^{1/2}}{4k^2L^{1/2}} = \frac{L}{2k} \) and \( \frac{-w_K}{w_L} = -\frac{2}{1} \), condition \( \text{TRS} = \frac{-w_K}{w_L} \Rightarrow \frac{-L}{2k} = -2 \Rightarrow (L = 4k).

**Step 2:** Relate \( K \) and \( y \) to get how much of each is required to produce \( y \). (So we can plug this into \( TC(y) \) in step 3): Since \( L = 4k \), \( yF(k, L) = 8k^2y(4k)^{1/2} = 16k^3y \) \( \Rightarrow \frac{K}{(\frac{4}{16})^{1/2}} \Rightarrow \frac{L}{4(\frac{4}{16})^{1/2}} \) (since \( L = 4k \)).

**Step 3:** From step 2 into the cost function \( c(y) = w_LL + w_KK \):

\[ c(y) = 1 \cdot 4(\frac{4}{16})^{1/2} + 2(\frac{4}{16})^{1/2} = 6(\frac{4}{16})^{1/2} \].

Since the cost function is convex (symmetric) we know, consistent with part (a), we have DRS.

**Exercise 2: Equilibrium with \( N \) Firms.**

A firm has a fixed cost of \( FC = 100 \) to build a factory in addition to variable cost \( VC(y) = 5y + y^2 \) to produce \( y \). The market demand for the good is \( D(p) = 1,000 - 10p \). There are 5 identical firms and all firms are price takers.

• (a) Find each firm’s supply curve.

- First, note that \( MC(y) = C'(y) = 5 + 2y \) and \( ATC(y) = \frac{TC(y)}{y} = \frac{6y^2 + 10}{y} \). The firm’s supply curve is such that \( p = MC(y) \) \( \Rightarrow p = 5 + 2y \Rightarrow y = \frac{p - 5}{2} \) for prices above the minimum of the \( ATC(y) \) curve (also known as \( \text{ATC}^{\text{mes}} \)), and \( y = 0 \) below this.

- Finding \( \text{ATC}^{\text{mes}} \): For \( y \) is \( \text{ATC}(y) = 0 \Rightarrow \frac{6y^2 + 10}{1} + 5 = 10 \Rightarrow \text{ATC}^{\text{mes}} = \frac{6y^2}{1} + 5 = 25 \). Each firm’s supply curve is \( y(p) = \frac{p - 5}{2} \) if \( p \geq 25 \) and \( y(p) = 0 \) if \( p < 25 \).

• (b) Find the aggregate supply in this market.

\[
\text{Aggregate supply is } S(p) = N \cdot y(p) = 5 \cdot \begin{cases} 0 & \text{for } p < 25 \\ \frac{p - 5}{2} & \text{for } p \geq 25 \end{cases}
\]

\[
S(p) = \begin{cases} & 5 & \text{for } p < 25 \\ & 5 \left( \frac{p - 5}{2} \right) & \text{for } p \geq 25 \end{cases}
\]
• (c) Find the equilibrium price and quantity in this market. What is the production level and profit for each firm?

  To find equilibrium price and quantity, we set \( S(p) = D(p) \) and solve for \( p \) and \( q \):
  \[ S(p) = D(p) \Rightarrow 5 \left( \frac{p-5}{3} \right) = 1000 - 10p \]
  (for \( p \geq 25 \)) and we get \( p^* = 25 \). Plug \( p = S(p) \) or \( D(p) \)) to get \( S(25) = 190 = y_{\text{max}}^* \)

  Each (identical) firm produces \( \frac{y_{\text{max}}^*}{N} = \frac{190}{5} = 38 = y^*_\text{firm} \).

  Each firm's profit is \( \pi = p^* y_{\text{firm}} - TC(y_{\text{firm}}) = 81(38) - (100 + 5(38) + 81^2) \)
  \[ \Rightarrow \pi = 1,344 \]

• (d) Assume now there is free entry in this market (firms can enter or exit). Determine

  - (i) the profit of each firm,
  - (ii) the market price,
  - (iii) the output of each firm, and
  - (iv) the number of firms in the market.

  (i) In the long run, free entry will result in \( \pi = 0 \)

  (Thinking back to Econ 101 for this problem: As more firms enter, profit decreases. Once profit is zero, no firms enter or exit. If profit is less than zero, firms would exit, and would cease exiting once profit was zero.)

  (ii) Profit is zero at \( \text{ATC}_{\text{mes}} \), so in the long run we must have price \( p^* = \text{ATC}_{\text{mes}} = 25 \).

  (iii) At \( p^* = 25 \), demand is \( D(p) = 1000 - 10(25) = 750 \).

  (iv) Each firm produces \( y_{\text{mes}}^* = 10 \), so the number of firms \( N = 75 \).