Econ 690
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Homework #1

In the traditional models of finance, investors are "risk averse" because they have diminishing marginal utility. An implication of diminishing marginal utility is that the increase in utility from gaining an amount x is less than the loss in utility from losing an equal amount. So, risk averse investors would never want to take a fair bet where the probability of winning x and losing x are equal.

A newer view of portfolio choice is that asset demand arises from investors such as banks (or perhaps individual investors) that face constraints on their balance sheets.

1.   Suppose we have a bank that gets an amount of deposits, $D > 0$ at the beginning of a period. The bank can use those deposits to either make loans, L, that pay an interest rate of i_L, or they can hold reserves, R, that pay an interest rate of i_R. Assume $i_L > i_R$. The bank has no capital of its own, so the amount that it can invest is constrained by the amount of deposits: $L + R = D$

At the end of the period, the bank's assets are worth $(1+i_L)L + (1+i_R)R$. The bank chooses its portfolio of L and R to maximize its profits: $(1+i_L)L + (1+i_R)R - D$.

The bank faces a constraint, however, which is that its profits must be positive at the end of the period, so that it can pay back depositors. But it is costly to "liquidate" loans – the bank needs to sell off the loans, but the cost of selling off the loans increases with the amount of loans the bank has made. Assume the cost is given by $a\dfrac{L^2}{D}$ where $a > 0$.

So actually the bank chooses L and R to maximize $(1+i_L)L + (1+i_R)R - D - a\dfrac{L^2}{D}$, subject to the constraint that $L + R = D$. We can substitute the constraint into the profit function to get:

$$W = (1+i_L)L + (1+i_R)(D-L) - D - a\frac{L^2}{D}.$$

Find the value of L that maximizes this expression. Then verify, after solving, that $W > 0$, so that the bank can indeed repay its depositors.
Note: We can allow $R < 0$, which means $L > D$. This means that the bank not only uses deposits but also borrows at an interest rate i_R and uses the proceeds to invest in loans that earn $i_L > i_R$.

2.   Now the set-up is the same as the problem above, but the return on loans is uncertain. With probability p, the loan earns i_G and with probability $1-p$ the loan earns i_B, where $0 < i_B < i_R < i_G$. Reserves pay an interest rate i_R with certainty. Assume $pi_G + (1-p)i_B > i_R$, so the expected return on loans is greater than the return on reserves.

Now the bank maximizes expected profits:

$$E(W) = p(1+i_G)L + (1-p)(1+i_B)L + (1+i_R)(D-L) - D - a\frac{L^2}{D}.$$

Find the value of L that maximizes the above expression. Let L^O be the value of L that maximizes this expression.

However, it is not enough that the banks expected profits are positive. Their profits need to be positive whether the loan earns i_G or i_B. This means profits have to be positive even when the outcome of the loan is "bad" and the bank earns only i_B.

State the conditions under which profits are positive when the outcome is bad for the value L^O that you solved above. In other words, under what conditions do we have

$$(1+i_B)L^O + (1+i_R)(D-L^O) - D - a\frac{(L^O)^2}{D} > 0 ?$$

If that condition is not satisfied for $L = L^O$ (profits would be negative if the bank chose $L = L^O$, which is not permissible), then what value of L is optimal for the bank?

3.   For comparison's sake, now consider the traditional finance model of portfolio choice. Here we will assume that investors have a concave utility function over their consumption. Assume that the investor puts his own cash, D, at risk. In other words, we are not considering a bank that takes deposits, but instead an investor that initially has wealth equal to D. This investor does not face any costs of liquidating loans, and has a utility function given by $U = W - aW^2$, where W is the end-of-period value of assets.

The investor's utility when the loans have a high return is given by:

$$U_G = (1+i_G)L + (1+i_R)(D-L) - a\left((1+i_G)L + (1+i_R)(D-L)\right)^2$$

and utility when returns on the loan are low is given by:

$$U_B = (1+i_B)L + (1+i_R)(D-L) - a\left((1+i_B)L + (1+i_R)(D-L)\right)^2.$$

Expected utility is $EU = pU_G + (1-p)U_B$. Find the choice of L that maximizes expected utility.

[Note: The quadratic utility function above is sort of odd because there is a maximum amount of consumption, above which the investor would be less happy if he had more consumption. Let's rule that case out. Marginal utility, $\dfrac{dU}{dW}$, is equal to $1 - 2aW$. Let's evaluate the investor's wealth at the end of the period at the level it would be if he put all of his wealth into the risky loans and he got the good return, so $W = (1 + i_G)D$. That means $\dfrac{dU}{dW} = 1 - 2a(1 + i_G)D$. Assume $1 - 2a(1 + i_G)D > 0$, so his marginal utility is positive. This condition implies that we would also have $1 - 2a(1 + i_R)D > 0$, and you will see that this assumption will give you that the amount of L the investor holds is positive.]