

Econ 690

Portfolio Choice

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We will examine the portfolio choice of an investor who wants a high expected return on his portfolio, but dislikes variance. He trades off expected return against variance.

We will then incorporate that model of asset choice into general equilibrium models that determine the return on assets. We derive the famous CAPM (Capital Asset Pricing Model.)

Portfolio Choice

Investor i wants to choose a portfolio this period to maximize:

$$E(W_{i,+1}) - \alpha_i \text{var}(W_{i,+1})$$

W_{+1} is the investor's wealth next period, which depends on the returns on his portfolio. He likes higher expected wealth next period, $E(W_{i,+1})$, but dislikes variance, $\text{var}(W_{i,+1})$.

The parameter α_i measures his dislike of variance compared to expected return. It is a measure of his “risk aversion”. More uncertainty increases the variance of next period's wealth.

The investor's wealth next period will be given by:

$$(1) \quad W_{i+1} = rX_{i,0} + r_1X_{i,1} + r_2X_{i,2} + \dots + r_nX_{i,n},$$

where X_i is the amount invested this period in each of the $n+1$ assets. The returns, r_i , are not known today when we make our investment.

(Asset 0 will take on particular importance below. For now, it is just another asset, whose return is r , but later we will assume r is the “riskless” asset whose return is known now.)

W_i is the investor's initial wealth – his wealth today.

If we let W equal today's wealth, the budget constraint for choosing the investment amounts is

$$W_i = X_{i,0} + X_{i,1} + X_{i,2} + \dots + X_{i,n}.$$

We can rewrite this constraint, dividing both sides of the equation by W :

$$(2) \quad 1 = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2} + \dots + \lambda_{i,n}.$$

Here, $\lambda_{i,j} = \frac{X_{i,j}}{W_i}$ is the share of initial wealth invested in asset j . Then we have:

$$W_{i+1} = W_i \left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n} \right).$$

Then we can see:

$$EW_{i,+1} = W_i E\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right), \text{ and}$$

$$\text{var}(W_{i,+1}) = W_i^2 \text{var}\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right).$$

The investor's goal is to maximize, therefore

$$(3) \quad \begin{aligned} & W_i E\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right) \\ & - \alpha_i W_i^2 \text{var}\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right). \end{aligned}$$

Maximizing this is the same as maximizing

(4)

$$E\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right) - a_i \text{ var}\left(r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right),$$

where $a_i \equiv \alpha_i W_i$, because W is known today, so dividing equation (3) by W_i does not change the optimal choices.

The investor then needs to choose the asset shares $\lambda_{i,j}$ to maximize equation (4) subject to the constraint (2).

Now let asset 0 be a deposit, such as a CD (certificate of deposit) that has no risk. It is a one-period asset, and there is no chance there will be default.

Therefore, we don't need to put the expectation sign in front of the return r since it is known. Also, r does not contribute to the variance of the return on the portfolio. We can simplify (4) as:

$$(5) \quad r\lambda_{i,0} + E(r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}) - a_i \text{var}(r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n})$$

Let's now define λ_i to be the sum of all of the investment shares on the risky assets:

$$(6) \quad \lambda_i = \sum_{j=1}^n \lambda_{i,j}.$$

Because the total shares have to add up to one, then we must have $\lambda_{i,0} = 1 - \lambda_i$.

Now define $\mu_{i,j}$ to be the share of the portfolio of risky assets only (the portfolio of assets that are comprised of assets 1 through n , whose returns are risky.) We have:

$$(7) \quad \mu_{i,j} \equiv \frac{\lambda_{i,j}}{\lambda_i}.$$

Let $r_{i,m}$ be the return on the investor's portfolio of risky assets:

$$(8) \quad r_{i,m} = r_1\mu_{i,1} + r_2\mu_{i,2} + \dots + r_n\mu_{i,n}.$$

Then we have

$$(9) \quad E(r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}) = \lambda_i E(r_1\mu_{i,1} + r_2\mu_{i,2} + \dots + r_n\mu_{i,n}) = \lambda_i E r_{i,m}.$$

Also, we have

$$(10) \quad \begin{aligned} \text{var}\left(r_1\lambda_{i,1} + r_2\lambda_{i,2} + \dots + r_n\lambda_{i,n}\right) &= \lambda_i^2 \text{var}\left(r_1\mu_{i,1} + r_2\mu_{i,2} + \dots + r_n\mu_{i,n}\right) \\ &= \lambda_i^2 \text{var}(r_{i,m}) \end{aligned}.$$

Substitute equations (9) and (10) into the objective, (5), and also substitute the constraint $\lambda_{i,0} = 1 - \lambda_i$, so that we can rewrite equation (5) as:

$$(11) \quad r(1 - \lambda_i) + \lambda_i E(r_{i,m}) - a_i \lambda_i^2 \text{var}(r_{i,m}).$$

We are breaking the optimization problem of the investor into two parts. First, he will choose λ_i , the share of his overall portfolio that will be invested in risky assets. The remainder, $1 - \lambda_i$ will be invested in the risk free deposit that pays a return r . Then he will choose the weights $\mu_{i,j}$ that each of the risky assets will get in his portfolio of risky assets.

The first-order condition for choosing λ_i to maximize (11) is:

$$-r + E(r_{i,m}) - 2a_i\lambda_i \text{var}(r_{i,m}) = 0,$$

which can be solved to give us:

$$(12) \quad \lambda_i = \frac{E(r_{i,m}) - r}{2a_i \text{var}(r_{i,m})}.$$

Holding the denominator constant, the investor puts a greater share into the risky portfolio when the return on the risky portfolio rises relative to the riskless return (when $E(r_{i,m}) - r$.)

He invests less in the risky portfolio, holding other things constant, when the variance of the risky portfolio, $\text{var}(r_{i,m})$, is higher. Comparing two investors who have identical risky portfolios, the investor with the higher degree of risk aversion (higher a_i) will invest a smaller share in the risky portfolio.

Now turn to the problem of choosing the optimal risky portfolio. The investor still wants to choose his portfolio to maximize the expression in equation (11). We can substitute our solution for the optimal value of λ_i into equation (11), and we find equation (11) can be rewritten as:

$$(13) \quad r + \frac{(E(r_{i,m}) - r)^2}{4a_i \text{var}(r_{i,m})}.$$

The investor will now choose the weights of his portfolio of risky assets, $\mu_{i,j}$, to maximize the expression in equation (13). However, the choice of variables to maximize any function also maximizes any linear function of the original function.

So the choice of $\mu_{i,j}$ that maximizes the expression in (13) also maximizes:

$$(14) \quad \frac{(E(r_{i,m}) - r)^2}{\text{var}(r_{i,m})}.$$

This expression is simply a linear function of the function in (13) (in which we subtract r from the expression in (13) and multiply by $4a_i$.)

We have actually just derived one of the most important results in investment theory. Look at equation (14). The investor wants to choose a risky portfolio – choose the values of the weights, $\mu_{i,j}$ - to maximize the expression in (14).

The choice of the $\mu_{i,j}$ that maximizes (14) does not depend on a_i ! All investors choose the same $\mu_{i,j}$. The choice of the $\mu_{i,j}$ does not depend on the investor's degree of risk aversion, a_i .

Suppose the world was full of risk-averse investor who only differed in their degree of risk aversion, a_i . All of these investors will choose exactly the same weights in their portfolio of risky assets. Why? Because they are all choosing their $\mu_{i,j}$ to maximize (14), but (14) does not depend on their degree of risk aversion. All investors choose $\mu_{i,j}$ to maximize exactly the same function.

The only way in which the degree of risk aversion will affect the portfolio choice is through its effect on λ_i , the share of wealth invested in risky assets.

As equation (12) shows, more risk averse investors put a smaller share of their wealth in the risky portfolio, and a greater share in the riskless asset.

Now consider the allocation of risky assets to maximize expression (13), keeping in mind that $r_{i,m}$ is given by equation (8). We can derive from the first-order condition for choosing $\mu_{i,j}$ the following relationship:

$$(15) \quad E(r_j) - r = \frac{E(r_m) - r}{\text{var}(r_m)} \text{cov}(r_j, r_m).$$

Notice that we dropped the subscript i from the return on the investor's risky portfolio – we've written r_m instead of $r_{i,m}$. That's because all investors choose the same risky portfolio,

Equilibrium Asset Returns

We investigate how asset returns are determined in an economy that is populated by investors that are like the ones in the previous section. The investors in the economy differ only in their degree of risk aversion, as captured in the parameter a_i .

We have already seen that the only way in which the investors' portfolios differ is in the division of their portfolio between the riskless asset and the risky portfolio. Investors' choice of λ_i , the share of their wealth invested in the portfolio of risky assets, depends on the degree of risk aversion. The larger is a_i , the smaller is λ_i .

We also noted that the allocation of each asset as a share of the risky portfolio is the same for all investors. The allocation does not depend on their degree of risk aversion. The μ_j from last section are the same for all investors.

Equation (15), derived from each investor's first-order conditions, is the same for all investors because they all choose the same risky portfolio. We repeat that equation here for convenience:

$$(16) \quad E(r_j) - r = \frac{E(r_m) - r}{\text{var}(r_m)} \text{cov}(r_j, r_m).$$

This equation can be considered an equilibrium relationship that determines the expected return on asset j .

We will refer to r_m as the return on the “market portfolio.” Since all investors hold the same risky portfolio, the weights of each asset in the risky portfolio must be equal to the value of that asset as a share of the value of all risky assets.

For example, if one percent of every investor’s risky portfolio is invested in Google stock, then one percent of risky investments for the economy as a whole must be in Google stock. So, Google’s share in the risky portfolio is equal to its value as a share of the value of all risky assets. We call it the market portfolio because the weight given to each asset is determined by the market value of that asset as a share of the total market value of risky assets.

A very common and almost famous way of writing equation (16) is:

$$(17) \quad E(r_j) - r = \beta_j (E(r_m) - r), \text{ where } \beta_j \equiv \frac{\text{cov}(r_j, r_m)}{\text{var}(r_m)}.$$

Among financial economists, β_j is called asset j 's "beta". That term is widely used on Wall Street.

The formula for β_j is the formula for the slope coefficient in a regression of r_j on r_m . Usually analysts measure r_m as the return on some broad index of equities, such as the S&P 500.

What is the intuition of this result? Equation (17) says that asset j has a higher expected return the larger is its β_j . Different assets have different betas depending on the covariance of the return on the asset with r_m . Assets that have a higher covariance with r_m have a higher expected return, which means the market perceives that asset as riskier.

The market insists on a higher expected rate of return in order to compensate investors for the risk they perceive in holding the asset. We can conclude that the market believes that the appropriate measure of risk is $\text{COV}(r_j, r_m)$.

To see why the market takes $\text{COV}(r_j, r_m)$ as the measure of the risk of asset j , write out the variance of the return on the market portfolio, r_m .

$$\text{var}(r_m) = \left(\begin{aligned} &\mu_1^2 \text{var}(r_1) + \dots + \mu_n^2 \text{var}(r_n) \\ &+ 2\mu_1\mu_2 \text{cov}(r_1, r_2) + \dots + 2\mu_1\mu_n \text{cov}(r_1, r_n) + 2\mu_2\mu_3 \text{cov}(r_2, r_3) \\ &+ \dots + 2\mu_{n-1}\mu_n \text{cov}(r_{n-1}, r_n) \end{aligned} \right)$$

If the investor increases its holdings of, for example, asset 1, we can calculate the effect on the variance:

$$\begin{aligned} \frac{d \text{var}(r_m)}{d\mu_1} &= 2\left(\mu_1 \text{var}(r_1) + \mu_2 \text{cov}(r_1, r_2) + \dots + \mu_n \text{cov}(r_1, r_n)\right) \\ &= 2\text{cov}(r_1, \mu_1 r_1 + \mu_2 r_2 + \dots + \mu_n r_n) \\ &= 2\text{cov}(r_1, r_m) \end{aligned}$$

We can see from this calculation that $\text{cov}(r_1, r_m)$ measures the influence of adding a little bit more of asset 1 to the variance of the return on the market portfolio.

An asset is considered risky by the market – and so deserves a higher expected return – if adding a little of that asset contributes to the overall variance of return on the portfolio. Since investors are averse to higher variance on the return to their portfolio, this is the appropriate measure of the riskiness of an asset.

Most assets have returns that tend to be positively correlated with r_m . When the overall economy falls into recession, the returns on stocks, housing, bonds and most other assets tends to decline, but they tend to rise together during booms. An asset is considered riskier the more it increases the variance of the portfolio.