Econ 702
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Answers to Homework 2

1. Suppose that the production function is the following:

$$Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha} + \gamma(K_t + cN_t)$$

It is assumed that $\gamma, c > 0$ and $0 < \alpha < 1$.

a. Assume the price of the good is one. Show the profit of a firm with the above production function equals zero.

Answer:
The profit function is given by $\Pi(K_t, N_t) = Y_t - w_t N_t - R_t K_t$. The First Order Conditions with respect to N_t and K_t are;

$$w_t = F_N(K_t, N_t) = (1-\alpha)K_t^{\alpha} N_t^{-\alpha} + \gamma c$$
$$R_t = F_K(K_t, N_t) = \alpha K_t^{\alpha-1} N_t^{1-\alpha} + \gamma$$

By plugging above conditions into the profit function, one can get

$$\Pi(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha} + \gamma(K_t + cN_t) - (1-\alpha+\alpha)K_t^{\alpha} N_t^{1-\alpha} - \gamma c N_t - \gamma K_t = 0$$

The profit of a firm is zero for any nonnegative values of K_t and N_t.

b. What are the degrees of homogeneity for the first partial derivatives of output with respect to K_t and N_t ($F_K(K_t, N_t)$ and $F_N(K_t, N_t)$)?

Answer:
Partial derivatives of output with respect to N_t and K_t are;

$$F_N(K_t, N_t) = (1-\alpha)K_t^{\alpha} N_t^{-\alpha} + \gamma c$$
$$F_K(K_t, N_t) = \alpha K_t^{\alpha-1} N_t^{1-\alpha} + \gamma$$

For any $x > 0$,
$$F_N(xK_t, xN_t) = (1-\alpha)(xK_t)^{\alpha}(xN_t)^{-\alpha} + \gamma c$$
$$= x^{\alpha-\alpha}(1-\alpha)(K_t)^{\alpha}(N_t)^{-\alpha} + \gamma c = F_N(K_t, N_t)$$

Similarly,
$$F_K(xK_t, xN_t) = \alpha(xK_t)^{\alpha-1}(xN_t)^{1-\alpha} + \beta\gamma c$$
$$= x^{\alpha-1+1-\alpha}\alpha(K_t)^{\alpha-1}(N_t)^{1-\alpha} + \gamma c = F_K(K_t, N_t)$$

Both partial derivatives are homogeneous degree 0.

c. What is the sign (positive, negative or zero) of the cross partial derivative of output with respect to K_t and N_t ($F_{KN}(K_t, N_t)$)?

Answer:

$$F_{KN}(K_t, N_t) = \alpha(1-\alpha)K_t^{\alpha-1}N_t^{-\alpha}$$

The cross partial derivatives are positive given any $0 < \alpha < 1$ and positive inputs.

d. Take the first partial derivative of the output per worker (Y_t/N_t) with respect to capital per worker (K_t/N_t). What value does the derivative converge to as $K_t/N_t \to \infty$?

Answer:

$\frac{Y_t}{N_t} = \left(\frac{K_t}{N_t}\right)^{\alpha} + \gamma(\frac{K_t}{N_t} + c)$. Then the partial derivative with respect to capital output ratio is

$$\frac{\partial(Y_t/N_t)}{\partial(K_t/N_t)} = \alpha\left(\frac{K_t}{N_t}\right)^{\alpha-1} + \gamma$$

Given $0 < \alpha < 1$, $\alpha(K_t/N_t)^{\alpha-1}$ converges to 0 as K_t/N_t goes to infinity. So $\frac{\partial(Y_t/N_t)}{\partial(K_t/N_t)} \to \gamma$ as $K_t/N_t \to \infty$.

2. Suppose that the production function is the following:

$$Y_t = AF(K_t, N_t) = A\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}$$

It is assumed that $\nu \geq 0$ and $0 < \alpha < 1$.

a. Prove that this production function features constant returns to scale.

Answer:

For any $x > 0$,

$$AF(xK_t, xN_t) = A\left[\alpha(xK_t)^{\frac{\nu-1}{\nu}} + (1-\alpha)(xN_t)^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}$$
$$= A\left[x^{\frac{\nu-1}{\nu}}\left\{\alpha(K_t)^{\frac{\nu-1}{\nu}} + (1-\alpha)(N_t)^{\frac{\nu-1}{\nu}}\right\}\right]^{\frac{\nu}{\nu-1}}$$
$$= x^{\frac{\nu-1}{\nu}\cdot\frac{\nu}{\nu-1}}A\left[\left\{\alpha(K_t)^{\frac{\nu-1}{\nu}} + (1-\alpha)(N_t)^{\frac{\nu-1}{\nu}}\right\}\right]^{\frac{\nu}{\nu-1}}$$

The last term equals $xAF(K_t, N_t)$. The production function is homogeneous degree 1, therefore features constant returns to scale.

b. What are signs (positive, negative or zero) of the first partial derivatives of output with respect to K_t and N_t ($AF_K(K_t, N_t)$ and $AF_N(K_t, N_t)$)?

Answer:

$$AF_K(K_t, N_t) = A\alpha\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}-1} K_t^{\frac{\nu-1}{\nu}-1}$$
$$= A\alpha\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{1}{\nu-1}} K_t^{-\frac{1}{\nu}}$$
$$= A\alpha\left[K_t^{\frac{1-\nu}{\nu}}\{\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\}\right]^{\frac{1}{\nu-1}}$$
$$= A\alpha\left[\alpha + (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{1}{\nu-1}}$$

So $AF_K(K_t, N_t) > 0$ for any $\nu \geq 0$, $0 < \alpha < 1$ and positive inputs.

Similarly, $AF_N(K_t, N_t) = A(1-\alpha)\left[\alpha\left(\frac{K_t}{N_t}\right)^{\frac{\nu-1}{\nu}} + (1-\alpha)\right]^{\frac{1}{\nu-1}}$ and this is also positive.

c. What are signs (positive, negative or zero) of own second partial derivatives of output with respect to K_t and N_t ($AF_{KK}(K_t, N_t)$ and $AF_{NN}(K_t, N_t)$)?

Answer:

$$AF_{KK}(K_t, N_t) = -\frac{A\alpha(1-\alpha)}{\nu}\left[\alpha + (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{1}{\nu-1}-1}\left(\frac{K_t}{N_t}\right)^{\frac{1-\nu}{\nu}-1}\frac{1}{N_t}$$

$$AF_{NN}(K_t, N_t) = -\frac{A\alpha(1-\alpha)}{\nu}\left[\alpha\left(\frac{N_t}{K_t}\right)^{\frac{1-\nu}{\nu}} + 1 - \alpha\right]^{\frac{1}{\nu-1}-1}\left(\frac{N_t}{K_t}\right)^{\frac{1-\nu}{\nu}-1}\frac{1}{K_t}$$

The above terms are both negative for given conditions on the parameters and positive inputs.

d. Express the capital per worker as a function of factor price ratio (w_t/R_t).

Answer:
The First Order Conditions with respect to N_t and K_t are;

$$w_t \; = \; AF_N(K_t, N_t) = A(1-\alpha)\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}-1} N_t^{\frac{\nu-1}{\nu}-1}$$

$$R_t \; = \; AF_K(K_t, N_t) = A\alpha\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}-1} K_t^{\frac{\nu-1}{\nu}-1}$$

Then the factor price ratio is

$$\frac{w_t}{R_t} = \frac{1-\alpha}{\alpha}\left(\frac{K_t}{N_t}\right)^{\frac{1}{\nu}}$$

The capital per worker expressed as a function of the factor price ratio is

$$\frac{K_t}{N_t} = \left\{\frac{\alpha}{1-\alpha}\frac{w_t}{R_t}\right\}^{\nu}$$

e. Under what conditions on the parameters is the labor share of output ($w_t N_t/Y_t$) is increasing or decreasing with the factor price ratio?

Answer:

$$\frac{w_t N_t}{Y_t} = \frac{A(1-\alpha)\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}-1} N_t^{\frac{\nu-1}{\nu}}}{A\left[\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}\right]^{\frac{\nu}{\nu-1}}}$$

$$= \frac{(1-\alpha)N_t^{\frac{\nu-1}{\nu}}}{\alpha K_t^{\frac{\nu-1}{\nu}} + (1-\alpha)N_t^{\frac{\nu-1}{\nu}}} = \frac{1-\alpha}{\alpha\left(\frac{K_t}{N_t}\right)^{\frac{\nu-1}{\nu}} + 1 - \alpha}$$

We already know the capital per worker is a function of the factor price ratio from part d. Using the result, one can get

$$\frac{w_t N_t}{Y_t} = \frac{1-\alpha}{\alpha\left(\frac{\alpha}{1-\alpha}\frac{w_t}{R_t}\right)^{\nu-1} + 1 - \alpha} = \frac{1}{\left(\frac{\alpha}{1-\alpha}\right)^{\nu}\left(\frac{w_t}{R_t}\right)^{\nu-1} + 1}$$

Partial derivative of labor share with respect to factor price ratio is

$$\frac{\partial\left(\frac{w_t N_t}{Y_t}\right)}{\partial\left(\frac{w_t}{R_t}\right)} = \frac{(1-\nu)\left(\frac{\alpha}{1-\alpha}\right)^{\nu}\left(\frac{w_t}{R_t}\right)^{\nu-2}}{\left\{\left(\frac{\alpha}{1-\alpha}\right)^{\nu}\left(\frac{w_t}{R_t}\right)^{\nu-1} + 1\right\}^2}$$

If $\nu > 1$, the above term is negative; labor share is decreasing with factor price ratio. If $\nu < 1$, the above term is positive; labor share is increasing with factor price ratio.