Econ 702
Spring 2020
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Homework 4

This problem set is meant to illustrate the importance of a government being able to commit to a policy. It proceeds in steps.

1. Suppose an individual lives for two periods. His utility function is

$$U = c_1 + \sqrt{c_2}$$

where c_1 is consumption in the first period and c_2 is consumption in the second period. At the beginning of period 1, he is born with income equal to y $(y > 1)$. He can either consume that income, or he can save it. If he saves it, he accumulates k capital. That is, $y - c_1 = k$.

Whatever he saves can be used to produce a consumption good in the second period, and he can consume whatever he produces. The production function is k, so $c_2 = k$.

We can substitute out for c_1 and c_2 from these equations and write utility as:

$$U = y - k + \sqrt{k} \ .$$

The individual takes y as given, and chooses k to maximize this expression.

Find the optimal value of k by taking the first-order condition for maximizing U in the above expression. Using your solution for k, and the equations $c_1 = y - k$ and $c_2 = k$, solve for c_1 and c_2 in terms of y. Then, using your solutions for c_1 and c_2, express utility, $U = c_1 + \sqrt{c_2}$, as a function of y.

Answer:
The FOC with respect to k is

$$\frac{\partial U}{\partial k} = -1 + \frac{1}{2} k^{-1/2} = 0$$

The optimal value of k is 1/4. This gives each period consumption as

$$c_1 = y - k = y - 1/4,\ c_2 = k = 1/4$$

Then the utility in terms of y is

$$U = y - k + \sqrt{k} = y - 1/4 + 1/2 = y + 1/4$$

2. This problem is exactly like the previous one, except that in period 2, the government will impose a tax on saving equal to *t*. Assume $0<t<1$. The individual knows at time 1 what the tax rate will be at time 2. He will have only an amount $(1-t)k$ available to produce output in period 2.

We have $c_1 = y-k$ but now, $c_2 = (1-t)k$.

We can write utility as:

$$U = y-k+\sqrt{(1-t)k}$$

Find the optimal value of *k* by taking the first-order condition for maximizing *U* in the above expression. Solve for *k* as a function of the variable *t*. (Let's call this function $g(t)$, for reference in problem 5 below.)

Using your solution for *k*, and the equations $c_1 = y-k$ and $c_2 = (1-t)k$, solve for c_1 and c_2 in terms of *y* and *t*.

Then, using your solutions for c_1 and c_2, express utility, $U = c_1+\sqrt{c_2}$, as a function of *y* and *t*.

Answer:

The FOC with respect to *k* is

$$\frac{\partial U}{\partial k} = -1+\frac{1}{2}(1-t)^{-1/2}k^{-1/2}(1-t) = -1+\frac{1}{2}(1-t)^{1/2}k^{-1/2} = 0$$

The optimal value of *k* in terms of *t* is

$$k = \frac{1-t}{4}$$

Using this, one can get c_1, c_2 and *U* as a function of *y* and *t*.

$$c_1 = y-\frac{1-t}{4}, c_2 = \frac{(1-t)^2}{4}, U = y-\frac{1-t}{4}+\frac{1-t}{2} = y+\frac{1-t}{4}$$

3. From question 2, solve for the government's tax revenue, $R = tk$, as a function of *t*, where *R* stands for revenue. That is, in question 2, you solved for *k* as a function of *t*. Now simply write the expression for revenue, $R = tk$, as a function of *t*.

Answer:

Plugging optimal *k* from question 2 into the revenue function gives the following

$$R = kt = \frac{t(1-t)}{4}$$

4. Now suppose the government at time 1 announces the tax rate it will set. Its objective is to maximize the sum of utility and the triple of its tax revenue, $U+3R$. Take your expression for utility from question 2, as a function of *y* and *t*, and revenue from question 3, as a function of *t*. Add them together to get $U+3R$. Find the first-order condition for choosing *t* to maximize $U+3R$. Solve for the optimal tax rate, *t*, from that first-order condition.

Answer:
The government maximizes the following objective

$$U+3R=c_1+\sqrt{c_2}+3tk=y-\frac{1-t}{4}+\sqrt{\frac{(1-t)^2}{4}}+\frac{3t(1-t)}{4}=y+\frac{1-t}{4}+\frac{3t(1-t)}{4}$$

The FOC with respect to *t* is

$$\frac{\partial(U+3R)}{\partial t}=-\frac{1}{4}+\frac{3}{4}(1-t)-\frac{3}{4}t=0$$

The solution for the optimal tax rate *t* is 1/3.

5. Suppose now instead of setting the tax rate at time 1, the government sets the tax rate at time 2. Now the government does not care about the past. It only wants to maximize the sum of current utility and the triple of tax revenue, $\sqrt{c_2}+3R$. The government takes household saving *k* as given to solve for optimal tax rate. That is, take $c_2=(1-t)k$ and $R=tk$. Add them together to get $\sqrt{(1-t)k}+3tk$. Find the optimal value of *t* by taking the first-order condition for maximizing $\sqrt{(1-t)k}+3tk$. Solve for *t* in terms of *k*.

You now have solved for *t* as some function of *k*, call it $f(k)$. Assume the households can figure out what tax rate the government will set in period 2 when they make their investment decision in period 1. Take your solution for the household's choice of *k* as a function of *t*, which we called $g(t)$, and replace *k* in the function $f(k)$ so that we get $t=f(g(t))$. You should get a quadratic equation for *t*. Solve for the optimal tax rate, *t* (The solutions for a quadratic equation of the form $at^2+bt+c=0,(a\neq 0)$ are $t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$). Among the two solutions you get, which value of *t* is the only feasible one? (Hint: Drop one of solutions for *t* which is greater than one, making c_2 negative.)

Answer:
The government maximizes the following objective

$$\sqrt{c_2}+3R=\sqrt{(1-t)k}+3tk$$

The FOC with respect to *t* is

$$\frac{\partial(\sqrt{c_2}+3R)}{\partial t}=-\frac{1}{2}(1-t)^{-1/2}k^{1/2}+3k=0$$

The optimal *t* in terms of *k* is

$$t=f(k)=1-\frac{1}{36k}$$

We know the household's optimal saving decision as a function of *t* from question 2.

$$k=g(t)=\frac{1-t}{4}$$

Now plug this *k=g(t)* into *t=f(k)*. Then one can get

$$t = f(g(t)) = 1 - \frac{1}{9(1-t)}$$

Multiplying $9(1-t)$ on both sides of the above equation and rearranging give the following quadratic equation for *t*.

$$9t^2 - 18t + 8 = 0$$

The solutions for the above equation are

$$t = \frac{18 \pm \sqrt{36}}{18} = \frac{3 \pm 1}{3}$$

A solution for *t* is 4/3 or 2/3. 4/3 is not a feasible solution; if *t* is greater than one, $c_2 = (1-t)k$ goes negative. Therefore, the only feasible solution for tax rate is 2/3.

6. Finally, compare your answers for the tax rate in question 4 to question 5. Is the tax rate higher when the government can commit to the tax rate ahead of time in period 1 (as in question 4) or when it cannot commit (as in question 5)? In which case is saving higher?

Answer:

Tax rate and saving when the government can commit are

$$t = 1/3, k = \frac{1-t}{4} = \frac{2}{12}$$

Tax rate and saving when the government cannot commit are

$$t = 2/3, k = \frac{1-t}{4} = \frac{1}{12}$$

Tax rate is higher when the government cannot commit to it and the saving is higher when the government can commit to the tax rate.

Bonus: Show that utility ($c_1 + \sqrt{c_2}$) is higher under commitment.

Answer:

Utility under commitment is

$$c_1 + \sqrt{c_2} = y - k + \sqrt{(1-t)k} = y - k + \frac{(1-t)}{2} = y + \frac{1-t}{4} = y + \frac{2}{12}$$

Utility without commitment is

$$c_1 + \sqrt{c_2} = y - k + \sqrt{(1-t)k} = y - k + \frac{(1-t)}{2} = y + \frac{1-t}{4} = y + \frac{1}{12}$$

The household utility is higher under commitment.