Econ 702
Spring 2020
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Answers to Homework 5

1. Consider a two-period model where a household chooses consumption C_t and C_{t+1}, and saving S_t, given the endowment Y_t and Y_{t+1} and interest rate r_t to maximize following lifetime utility $U = u(C_t) + bu(C_{t+1})$ with $u(C) = -\exp(-aC)$, $\beta \in (0,1)$, $\alpha > 0$.

   (a) Note that $u(C) < 0$ for all consumption levels *C*. Why don't we need to worry about negative values of utility?

   Answer: That's because utility is an ordinal concept. What's important is that a high value of u is better than low value.

   (b) Show that the utility function $u(C)$ features a positive marginal utility and a diminishing marginal utility.

   Answer: $u'(C) = a\exp(-aC) > 0$ and $u''(C) = -a^2\exp(-aC) < 0$. Thus $u(C)$ shows a positive marginal utility and a diminishing marginal utility.

   (c) Write down the household problem for finding the optimal C_t and C_{t+1}. Derive the first-order conditions. Then solve for C_t as a function of Y_t, Y_{t+1}, r_t (and, of course, the parameters b, a).

   Answer: Household maximizes the lifetime utility
$$U = -\exp(-aC_t) - b\exp(-aC_{t+1})$$
given the lifetime budget constraint
$$C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t}.$$
As we learned in class, we can substitute for C_{t+1} in the utility function. Then the household's utility becomes
$$\square\ U = -\exp(-aC_t) - b\exp\left(-a(Y_{t+1} + (1+r_t)(Y_t - C_t))\right)$$
And the first order condition is
$$a\exp(-aC_t) - ab(1+r_t)\exp\left(-a(Y_{t+1} + (1+r_t)(Y_t - C_t))\right) = 0.$$
Since $C_{t+1} = Y_{t+1} + (1+r_t)(Y_t - C_t)$, we can write
$$a\exp(-aC_t) - ab(1+r_t)\exp\left(-aC_{t+1}\right) = 0 \Leftrightarrow \exp(-aC_t) = b(1+r_t)\exp\left(-aC_{t+1}\right)$$
By taking log, we have

$$-\alpha C_t = \ln(\beta(1+r_t)) - \alpha C_{t+1} \Leftrightarrow C_{t+1} = C_t + \frac{1}{\alpha}\ln(\beta(1+r_t))$$

The we can substitute this for C_t in the lifetime budget constraint.

$$C_t + \frac{C_t + \frac{1}{\alpha}\ln(\beta(1+r_t))}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t}.$$

Rearranging, we have

$$C_t = \frac{1+r_t}{2+r_t}\left(Y_t + \frac{Y_{t+1}}{1+r_t} - \frac{1}{a(1+r_t)}\ln(b(1+r_t)) \right).$$

2. Consider a consumer with a lifetime utility function

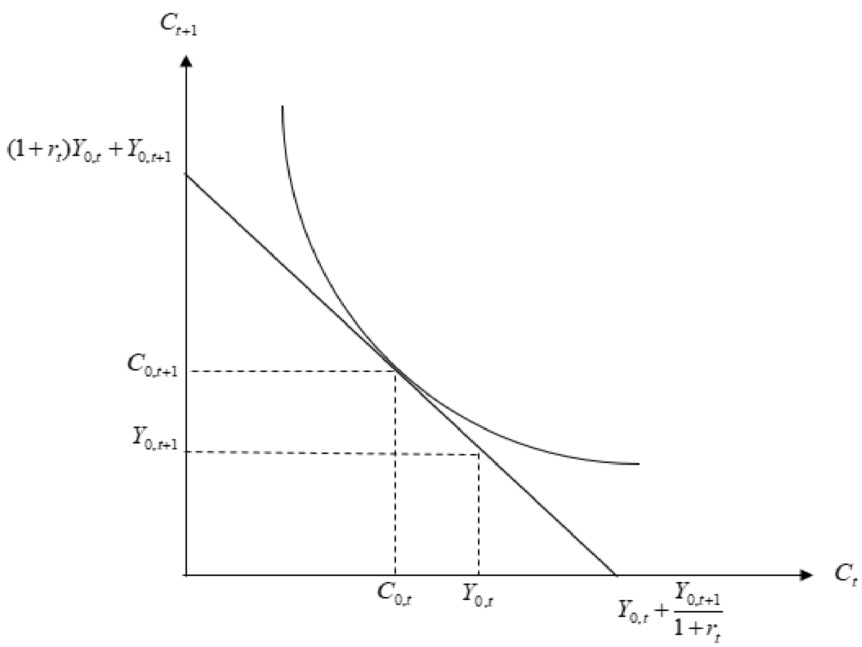
$$U = u(C_t) + bu(C_{t+1})$$

that satisfies all the standard assumption listed in lecture. The lifetime budget constraint is

$$C_t + \frac{C_{t+1}}{1+r_t} = Y_t + \frac{Y_{t+1}}{1+r_t} \quad .$$

(a) Suppose the first period optimal consumption C_t is less than Y_t, i.e. the consumer is saving at the first period. Graphically depict the optimality condition. Carefully label the intercepts of the budget constraint and optimal consumption points.

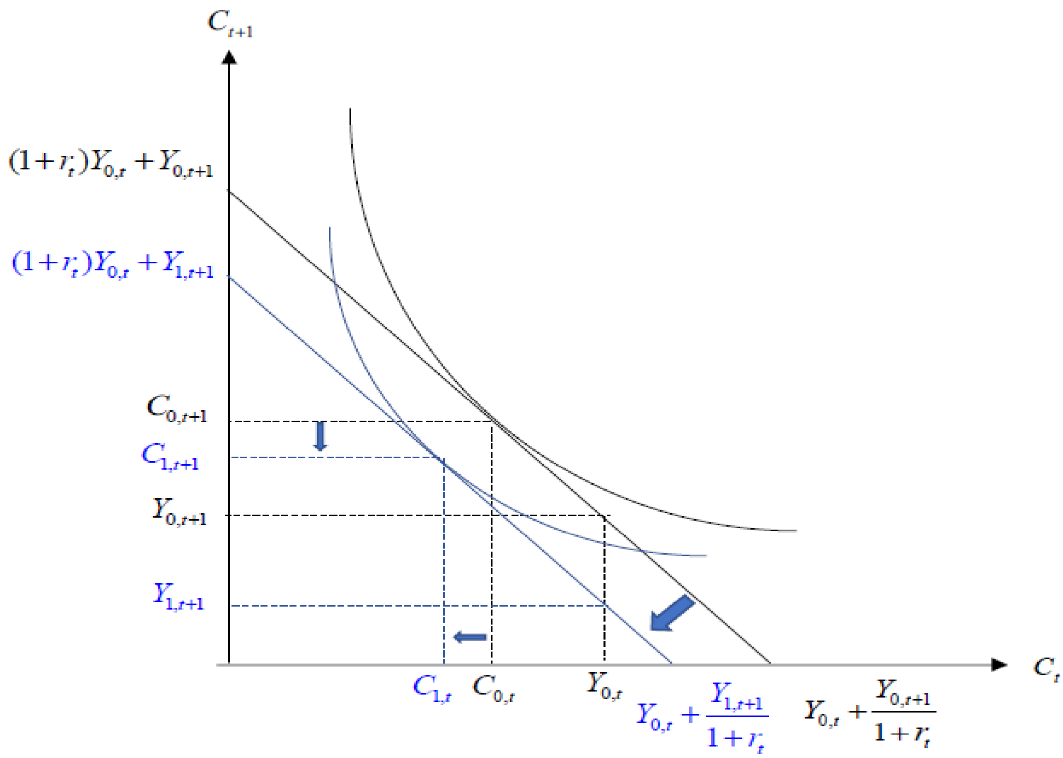
Answer:



where $C_{0,t}, C_{0,t+1}$ be optimal consumption bundle and $Y_{0,t}, Y_{0,t+1}$ be the endowments.

(b) Suppose there is a decrease in future income Y_{t+1}. Graphically depict the effects of a decrease in Y_{t+1}. Carefully label the intercepts of the budget constraint and optimal consumption points.

Answer:



where $C_{0,t}, C_{0,t+1}$ be optimal consumption bundle and $Y_{0,t}, Y_{0,t+1}$ be the endowments before the change and $C_{1,t}, C_{1,t+1}$ be optimal consumption bundle and $Y_{1,t+1}$ be the endowment after the change.

3. Prove that $\widetilde{C}_t < \frac{Y_t + E_t Y_{t+1}}{2}$ at page 11 in the lecture slide: Optimal Consumption and Saving, part 2. (Hint: In proving this, it is helpful to bring the term involving the square root to one side of the inequality, and everything else to the other side, then square both sides.)

Answer:

$$\widetilde{C}_t < \frac{Y_t + E_t Y_{t+1}}{2}$$

$$\Leftrightarrow \frac{3(Y_{t+1}^h + Y_{t+1}^l + 2Y)\sqrt{9(Y_{t+1}^h + Y_{t+1}^l + 2Y)^2 - 32(Y_{t+1}^h + Y)(Y_{t+1}^l + Y)}}{8} < \frac{Y_t + E_t Y_{t+1}}{2}$$

$$\Leftrightarrow 3(Y_{t+1}^h + Y_{t+1}^l + 2Y) - \sqrt{9(Y_{t+1}^h + Y_{t+1}^l + 2Y)^2 - 32(Y_{t+1}^h + Y)(Y_{t+1}^l + Y)} < 4(Y_t + E_t Y_{t+1})$$

$$\Leftrightarrow (Y_{t+1}^h + Y_{t+1}^l + 2Y) < \sqrt{9(Y_{t+1}^h + Y_{t+1}^l + 2Y)^2 - 32(Y_{t+1}^h + Y)(Y_{t+1}^l + Y)}$$

$$\Leftrightarrow (Y_{t+1}^h + Y_{t+1}^l + 2Y)^2 < 9(Y_{t+1}^h + Y_{t+1}^l + 2Y)^2 - 32(Y_{t+1}^h + Y)(Y_{t+1}^l + Y)$$

$$\Leftrightarrow 32(Y_{t+1}^h + Y)(Y_{t+1}^l + Y) < 8(Y_{t+1}^h + Y_{t+1}^l + 2Y)^2$$

$$\Leftrightarrow 0 < 8(Y_{t+1}^h)^2 - 16Y_{t+1}^h Y_{t+1}^l + 8(Y_{t+1}^l)^2$$

$$\Leftrightarrow 0 < 8(Y_{t+1}^h - Y_{t+1}^l)^2$$

Since $Y_{t+1}^h > Y_{t+1}^l$, $0 < 8(Y_{t+1}^h - Y_{t+1}^l)^2$ is satisfied so that $\widetilde{C}_t < \frac{Y_t + E_t Y_{t+1}}{2}$ is satisfied.