Econ 702

Macroeconomics I

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Spring 2020

Lecture 2: Solow Growth Model I

We would like to understand long-run sustained growth, and what might lead to poor countries "catching up" to rich countries.

We want a model that is consistent with these facts:

1. Output per worker grows at a roughly constant rate over long periods
2. Capital per worker grows at a roughly constant rate over long periods
3. Capital/output ratio is roughly constant over long periods
4. Labor's share (wN/Y) is roughly constant over time
5. Rate of return to capital is roughly constant $R=(Y-wN)/K$
6. Real wages grow at roughly a constant rate over long periods

Cross-country facts:

1. Enormous variation across countries
2. There are growth miracles and growth disasters
3. Strong correlation between income per capita and human capital

Solow Growth Model

A model developed by Nobel prize winning economist Robert Solow in the 1950s.

In this model, saving, investment and capital accumulation play an important role.

Also total factor productivity, (TFP), which simply is a term for changes in output that occur holding factor inputs constant.

The model is based on assuming a production function for a "representative" firm, that uses capital and labor as inputs to produce output.

Labor supply and TFP are taken as exogenous here.

Production function:

$$Y_t = A_t F(K_t, N_t)$$

Y_t is output, A_t is TFP, K_t is capital input, N_t is labor input.

We assume the production function is constant returns to scale, or "homogenous of degree one". Why?

This implies the firm size will not be determined in the model. Why?

This implies firms make zero profit. Intuitively, why?

Derive this mathematically

Firms rent capital at a rate r_t, and pay workers a wage of w_t. Those variables are exogenous to the firm (but endogenous in the economy.)

It chooses K_t and N_t to maximize

$$\Pi_t = AF(K_t, N_t) - w_t N_t - R_t K_t$$

(Notice the price of the good is equal to one.)

The 1st-order conditions are:

$$w_t = AF_N(K_t, N_t)$$

$$R_t = AF_K(K_t, N_t).$$

A function is said to be homogenous of degree ρ if:

$$F(\gamma K_t, \gamma N_t) = \gamma^\rho F(K_t, N_t)$$

Euler's theorem states for a function that is homogenous of degree ρ:

$$\rho F(K_t, N_t) = F_K(K_t, N_t)K_t + F_N(K_t, N_t)N_t$$

We've assumed $\rho = 1$. so

$$Y_t = A_t F(K_t, N_t) = A_t F_K(K_t, N_t)K_t + A_t F_N(K_t, N_t)N_t$$

But from the first-order conditions, we get:

$Y_t = R_t K_t + w_t N_t$, so profits are zero.

Important Properties of Production Functions

We will assume $F_K > 0$ and $F_N > 0$. (Positive marginal product.)

We will assume $F_{KK} < 0$ and $F_{NN} < 0$ (Diminishing marginal product)

We will assume $F_{KN} > 0$

We will assume $F\left(K, 0\right) = 0$ and $F\left(0, N\right) = 0$

Cobb-Douglas Example:

$$F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}, \quad \text{with } 0 < \alpha < 1.$$

The marginal products of capital and labor are given by:

$$F_K(K_t, N_t) = \alpha K_t^{\alpha-1} N_t^{1-\alpha}$$

$$F_N(K_t, N_t) = (1-\alpha) K_t^{\alpha} N_t^{-\alpha}$$

We can see $F_K > 0$ and $F_N > 0$.

We also have:

$$F_{KK}(K_t, N_t) = \alpha(\alpha - 1) K_t^{\alpha-2} N_t^{1-\alpha}$$

$$F_{NN}(K_t, N_t) = -\alpha(1-\alpha) K_t^{\alpha} N_t^{-\alpha-1}$$

$$F_{KN}(K_t, N_t) = (1-\alpha)\alpha K_t^{\alpha-1} N_t^{-\alpha}.$$

We can see $F_{KK} < 0$, $F_{NN} < 0$, and $F_{KN} > 0$.

The Cobb-Douglas production function is homogenous of degree one:

$$\begin{aligned} F(\gamma K_t, \gamma N_t) &= (\gamma K_t)^{\alpha}(\gamma N_t)^{1-\alpha} \\ &= \gamma^{\alpha} K_t^{\alpha} \gamma^{1-\alpha} N_t^{1-\alpha} \\ &= \gamma K_t^{\alpha} N_t^{1-\alpha}. \end{aligned}$$

Also,

$$\begin{aligned} F(0, N_t) &= 0^{\alpha} N_t^{1-\alpha} \\ F(K_t, 0) &= K_t^{\alpha} 0^{1-\alpha}. \end{aligned}$$

So, $F(K,0)=0$ and $F(0,N)=0$.

Households

The household begins each period with its capital stock, K_t, which it can rent to firms. Each household is also endowed with N_t units of labor, which it supplies inelastically to firms.

The household uses its labor income for consumption, or for saving. ,It saves by accumulating capital, so its saving is equal to new investment in the economy. We assume that newly saved capital takes one period before it can be used in production.

The household budget constraint is given by:

$$C_t + I_t \leq w_t N_t + R_t K_t + \Pi_t$$

where Π_t are profits remitted to the households from firms.

But we have already seen that $\Pi_t = 0$, and we know $Y_t = r_t K_t + w_t N_t$, so we have $C_t + I_t = Y_t$. This is a closed economy with no government spending or taxes.

Household investment adds to next period's capital stock. But also, some of this period's capital stock depreciates:

$$K_{t+1} = I_t + (1 - \delta) K_t$$

The household must decide how much to consume and how much to save. Solow assumes that a constant fraction of income is saved:

$$I_t = sY_t$$

$$C_t = (1 - s) Y_t$$

We can write the equations of the Solow model as:

$$Y_t = AF(K_t, N_t)$$

$$Y_t = C_t + I_t$$

$$K_{t+1} = I_t + (1 - \delta)K_t$$

$$I_t = sY_t$$

$$w_t = AF_N(K_t, N_t)$$

$$R_t = AF_K(K_t, N_t).$$

Given that N_t and A_t are exogenous, if we know how much capital we enter period t with, we can solve for the endogenous variables, Y_t, C_t, I_t, w_t and R_t.

With a given initial capital stock, K_0, the equation of motion for capital should allow us to solve for all endogenous variables at all t.

It will be helpful to analyze the motion of capital per worker, $\dfrac{K_t}{N_t}$.

Since

$$K_{t+1} = sAF(K_t, N_t) + (1-\delta)K_t$$

we have

$$\frac{K_{t+1}}{N_t} = \frac{sAF(K_t, N_t)}{N_t} + (1-\delta)\frac{K_t}{N_t}$$

Define:

$$k_t \equiv K_t/N_t$$

Then we can use the constant-returns-to-scale assumption to get:

$$\frac{F(K_t, N_t)}{N_t} = F\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = F(k_t, 1)$$

Let's define

$$f(k_t) \equiv F(k_t, 1)$$

So, we can write

$$\frac{K_{t+1}}{N_t} = sAf(k_t) + (1-\delta)k_t$$

Then, multiply and divide by next period's number of workers:

$$\frac{K_{t+1}}{N_{t+1}}\frac{N_{t+1}}{N_t} = sAf(k_t) + (1-\delta)k_t$$

But for now we will assume

$$N_{t+1}/N_t = 1$$

so,

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t$$

This is a difference equation in capital per worker.

$$k_{t+1} = sAf(k_t) + (1-\delta)k_t$$

Discuss.

We cannot solve it explicitly, because we have not even specified the function *f*. We could solve it algebraically for the Cobb-Douglas case. Instead, we will mostly analyze it graphically.

We can solve for the other endogenous variables in per worker terms. Define $y_t \equiv \frac{Y_t}{N_t}$, $c_t = \frac{C_t}{N_t}$, and $i_t = \frac{I_t}{N_t}$. We have:

$$y_t = Af(k_t)$$

$$c_t = (1-s)Af(k_t)$$

$$i_t = sAf(k_t).$$

We can also derive

$$R_t = Af'(k_t)$$

$$w_t = Af(k_t) - k_t Af'(k_t)$$

To see this, first we can use this result for functions that are homogenous of degree ρ:

$$F_K(\gamma K_t, \gamma N_t) = \gamma^{\rho-1} F_K(K_t, N_t)$$

We have $\rho = 1$, so

$$F_K(K_t, N_t) = F_K\left(\frac{K_t}{N_t}, \frac{N_t}{N_t}\right) = f'(k_t).$$

Also,

$$F(K_t, N_t) = f'(k_t)K_t + F_N(K_t, N_t)N_t$$
$$f(k_t) = f'(k_t)k_t + F_N(K_t, N_t)$$
$$F_N(K_t, N_t) = f(k_t) - f'(k_t)k_t.$$

Using

$$w_t = AF_N(K_t, N_t)$$
$$R_t = AF_K(K_t, N_t).$$

we get

$$R_t = Af'(k_t)$$
$$w_t = Af(k_t) - k_t Af'(k_t)$$

Cobb-Douglas case

$$k_{t+1} = sAk_t^{\alpha} + (1 - \delta)k_t.$$

$$y_t = Ak_t^{\alpha}$$

$$c_t = (1 - s)Ak_t^{\alpha}$$

$$i_t = sAk_t^{\alpha}$$

$$R_t = \alpha Ak_t^{\alpha-1}$$

$$w_t = (1 - \alpha)Ak_t^{\alpha}.$$