Econ 702

Macroeconomics I

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Spring 2020

Lecture 3: Solow Growth Model II

The equations of the Solow growth model, in per-worker terms, are given by:

$$k_{t+1} = sAf(k_t) + (1 - \delta)k_t$$
$$y_t = Af(k_t)$$
$$c_t = (1 - s)Af(k_t)$$
$$i_t = sAf(k_t).$$

$$R_t = Af'(k_t)$$
$$w_t = Af(k_t) - k_t Af'(k_t)$$

Cobb-Douglas case

$$k_{t+1} = sAk_t^\alpha + (1-\delta)k_t.$$

$$y_t = Ak_t^\alpha$$

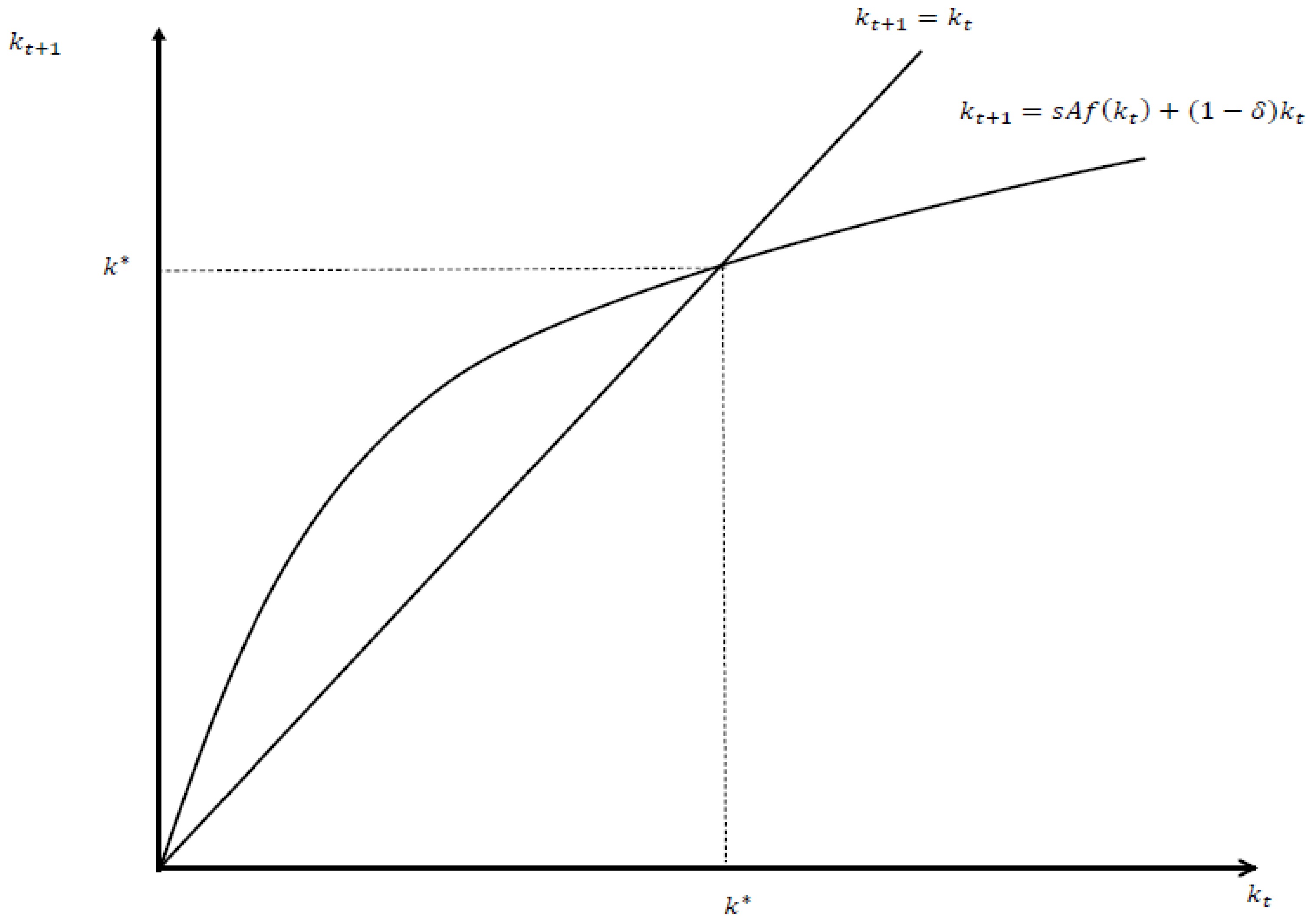
$$c_t = (1-s)Ak_t^\alpha$$

$$i_t = sAk_t^\alpha$$

$$R_t = \alpha Ak_t^{\alpha-1}$$

$$w_t = (1-\alpha)Ak_t^\alpha.$$

Graph k_{t+1} against k_t



Why does $k_{t+1} = sAf\left(k_t\right) + \left(1-\delta\right)k_t$ look like this?

Find the slope of the curve:

$$\frac{dk_{t+1}}{dk_t} = sAf'(k_t) + (1-\delta)$$

This slope is positive since $f' > 0$.

The slope is decreasing since $f'' < 0$.

We assume Inada conditions:

$$\lim_{k_t \to 0} f'(k_t) = \infty$$

$$\lim_{k_t \to \infty} f'(k_t) = 0.$$

The Inada conditions tell you the slope of the curve initially is infinite, and eventually falls to zero.

So the curve must eventually rise above the 45 degree line (the line where $k_{t+1} = k_t$.) But given our assumptions, it will cut it again from above one more time.

At the point where the curve cuts the 45 degree line, $k_{t+1} = k_t$. That is the "steady state". That is, at that point, $k_t = k_{t+1} = sAf\left(k_t\right) + \left(1-\delta\right)k_t$. The steady state level of capital per worker is determined by the equation

$$k^* = sAf\left(k^*\right) + \left(1-\delta\right)k^*$$

Notice that I dropped the time subscript here. Why?

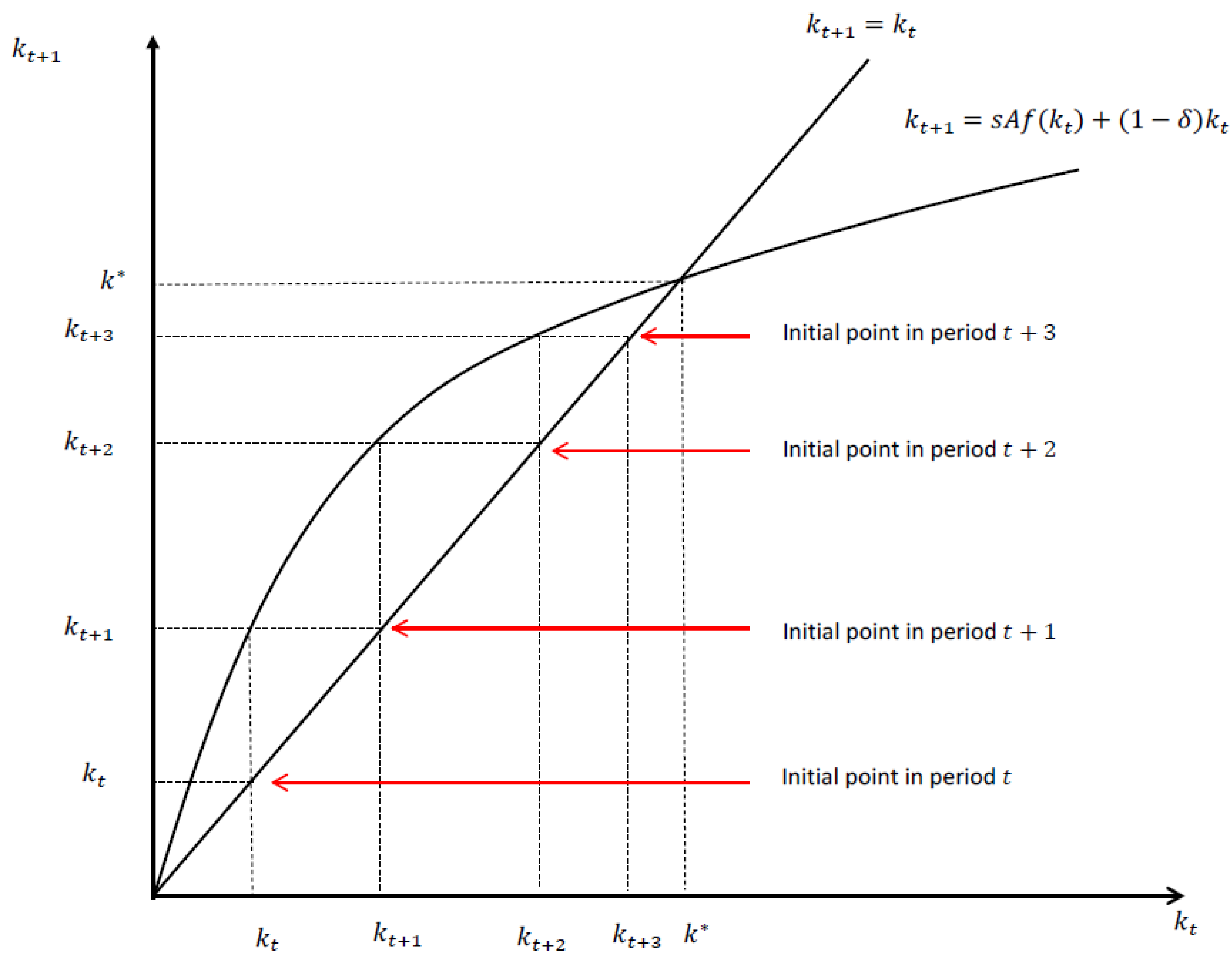
Convergence to the Steady State

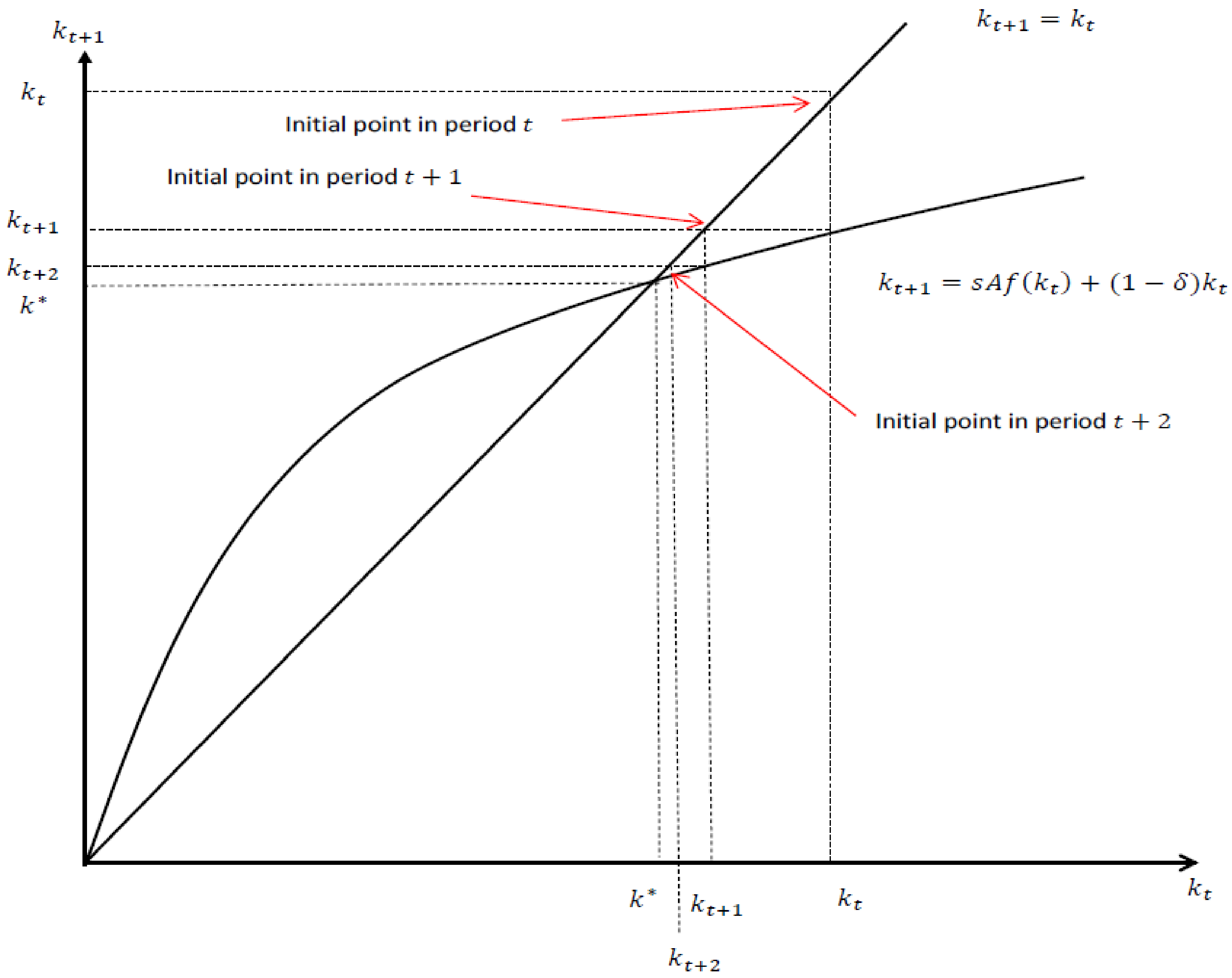
Do we converge to the steady state?

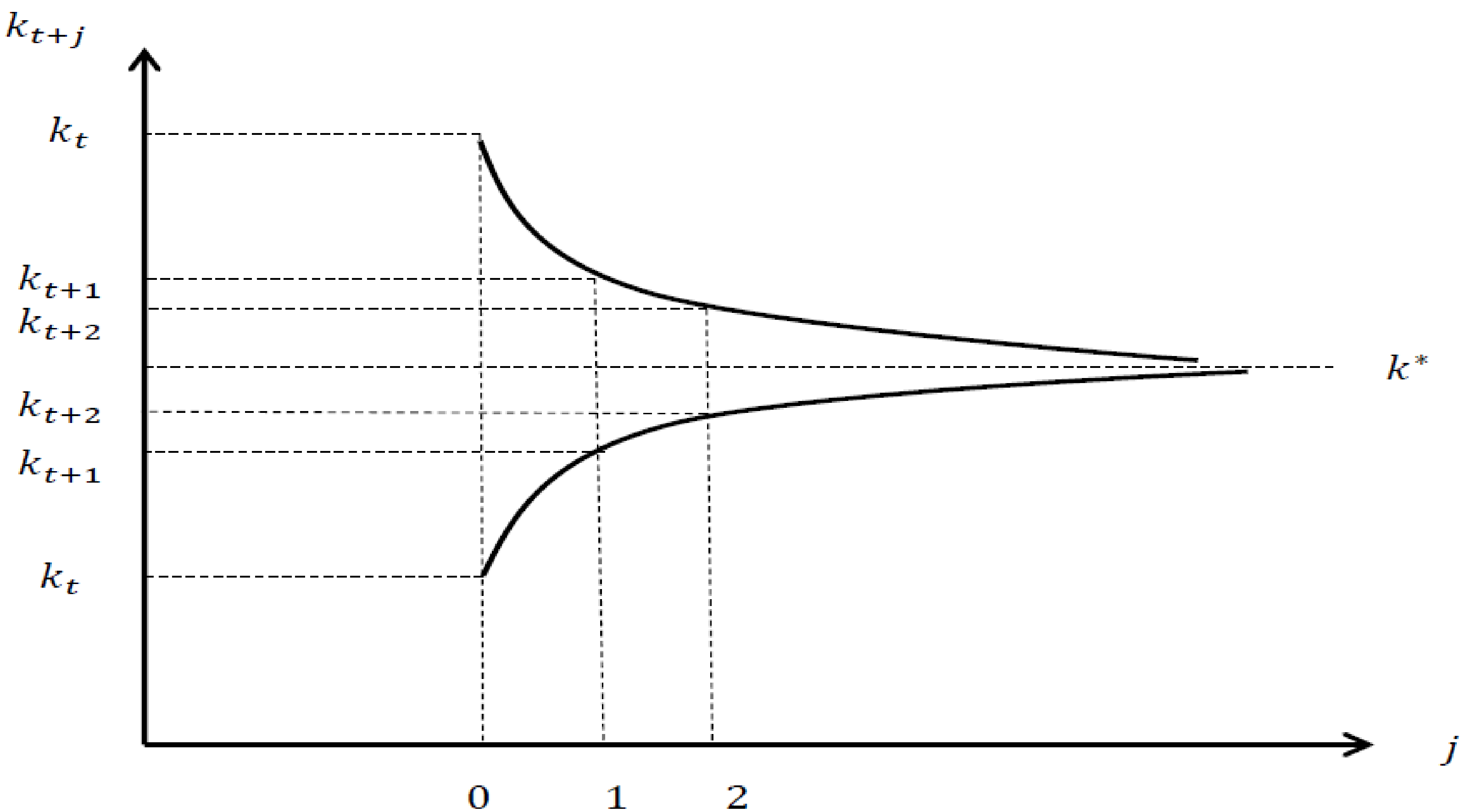
That is, if initial capital per worker at time t is such that $k_t < k^*$, does capital per worker increase toward k^*?

If initial capital per worker is such that $k_t > k^*$, does capital per worker fall toward k^*?

We can answer that mechanically on the next two graphs.


k_{t+1}
$k_{t+1} = k_t$
$k_{t+1} = sAf(k_t) + (1-\delta)k_t$
k^*
k_{t+3}
k_{t+2}
k_{t+1}
k_t
Initial point in period $t+3$
Initial point in period $t+2$
Initial point in period $t+1$
Initial point in period t
k_t
k_{t+1}
k_{t+2}
k_{t+3}
k^*
k_t


k_{t+1}
k_t
k_{t+1}
k_{t+2}
k^*
$k_{t+1} = k_t$
Initial point in period t
Initial point in period $t + 1$
$k_{t+1} = sAf(k_t) + (1 - \delta)k_t$
Initial point in period $t + 2$
k^*
k_{t+1}
k_t
k_t
k_{t+2}



What is the economics behind convergence to the steady state?

Why is there convergence at all? What conditions are needed for convergence?

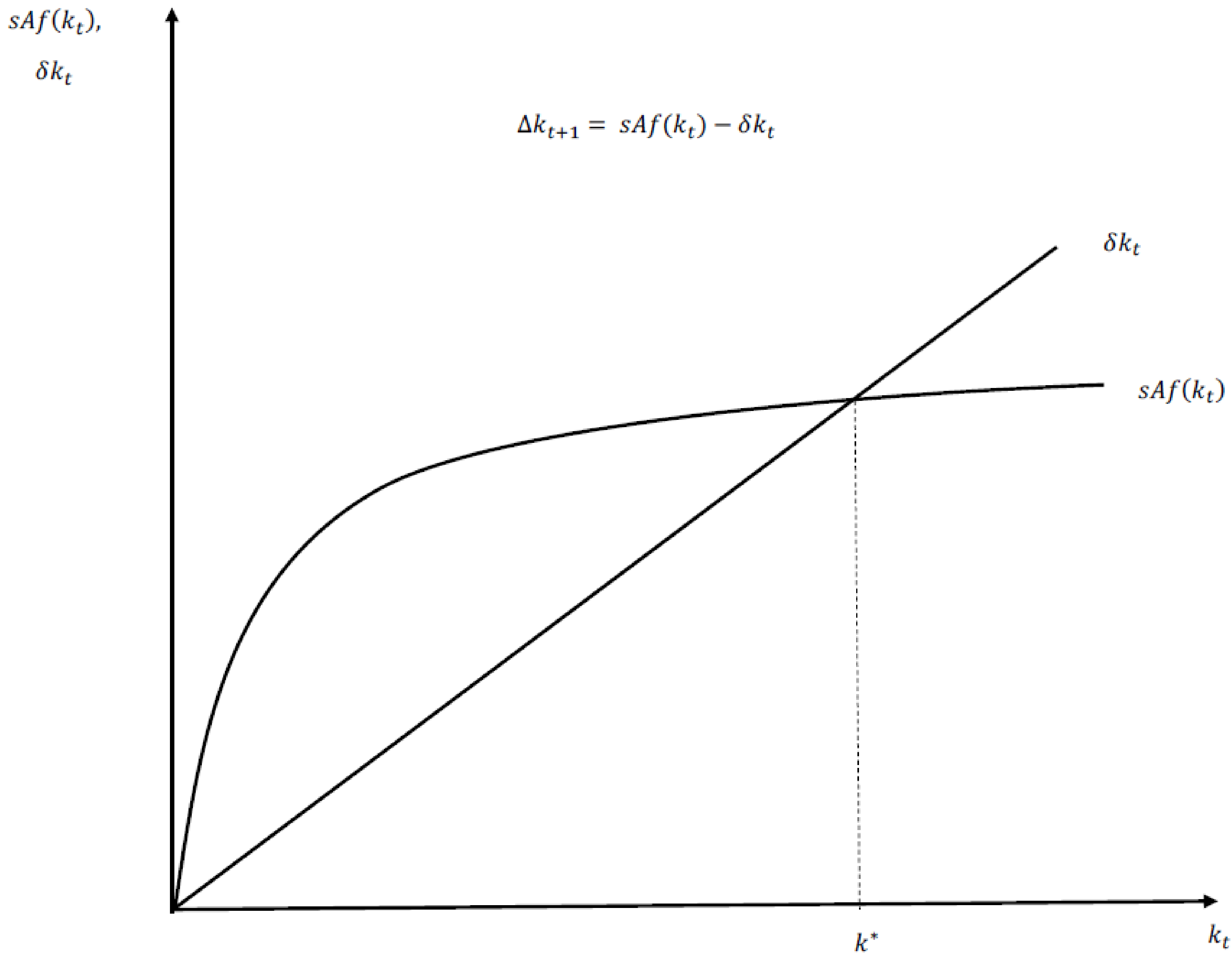
Why do we stay at a steady state?

Alternative graph may help us understand. This is the graph that Solow used and is in a lot of textbooks.

It graphs the new investment, $sAf\left(k_t\right)$, and the amount of investment required to replace depreciating capital, δk_t.

When new investment is greater than "required" investment, the capital stock increases. When it is lower than "required" investment, the capital stock falls.

At the steady state, new investment is just enough to replace depreciating capital.


$sAf(k_t)$,
δk_t
$\Delta k_{t+1} = sAf(k_t) - \delta k_t$
δk_t
$sAf(k_t)$
k^*
k_t

The steady state

We can see that in this version of the Solow model, if capital in a country is scarce initially, then capital per worker will increase.

In turn, because output per worker depends on capital per worker, it will also increase over time.

So, a country with scarce capital, $k_t < k^*$, will experience growth.

But the growth does not continue forever. Eventually output per worker approaches a steady state determined by capital per worker.

In understanding why countries have different levels of income per worker, it is then useful to understand what determines the steady state levels of capital per worker in the Solow model.

In the case of Cobb-Douglas production function, the equation $k^* = sAf\left(k^*\right) + \left(1-\delta\right)k^*$ is given by:

$$k^* = sAk^{*\alpha} + (1-\delta)k^*$$

The solution to this is:

$$k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$$

Higher saving, higher productivity (and lower depreciation) give a higher long-run capital per worker level.

We can also find:

$$y^* = Ak^{*\alpha}$$

$$c^* = (1-s)Ak^{*\alpha}$$

$$i^* = sAk^{*\alpha}$$

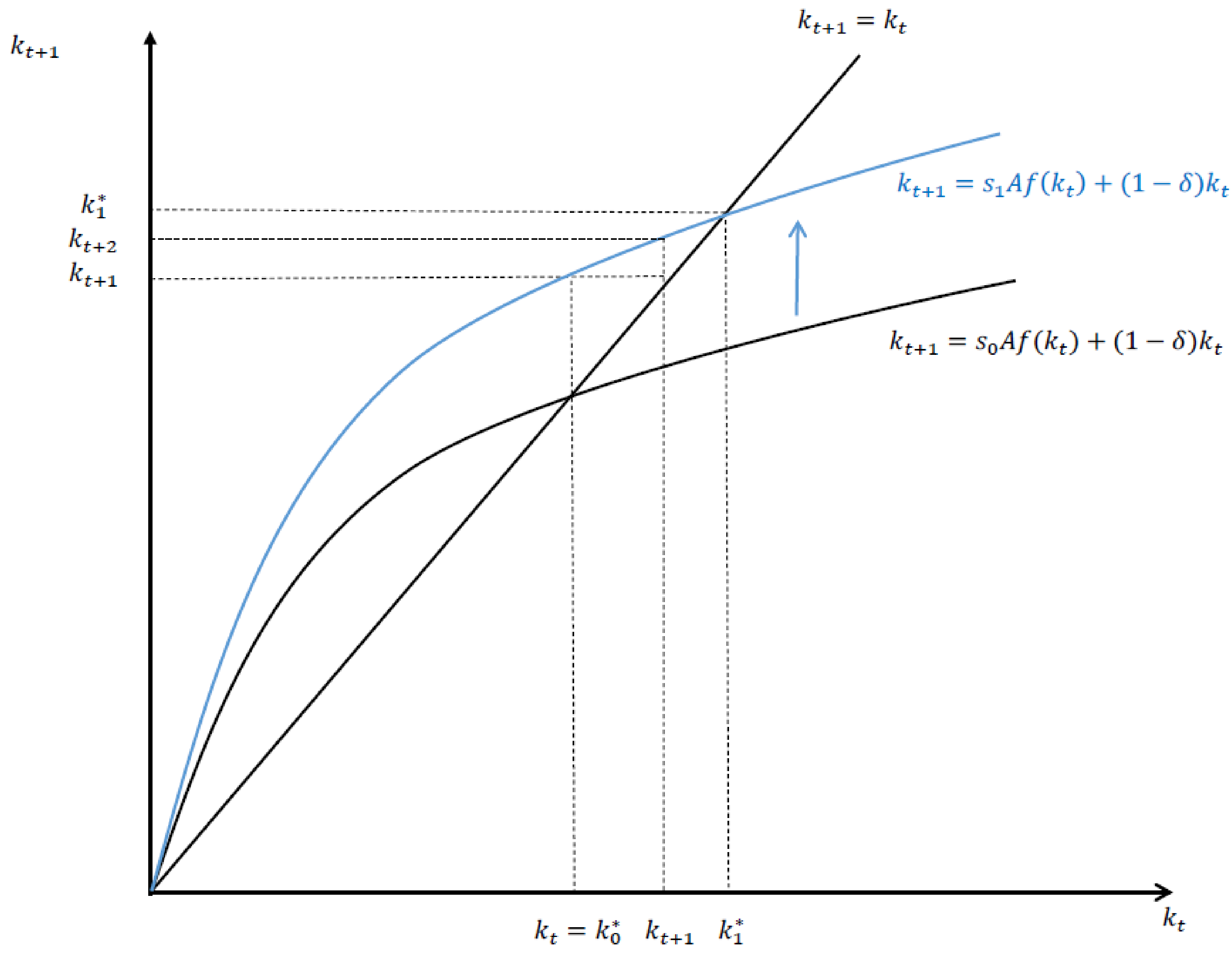
$$R^* = \alpha Ak^{*\alpha-1}$$

$$w^* = (1-\alpha)Ak^{*\alpha}$$

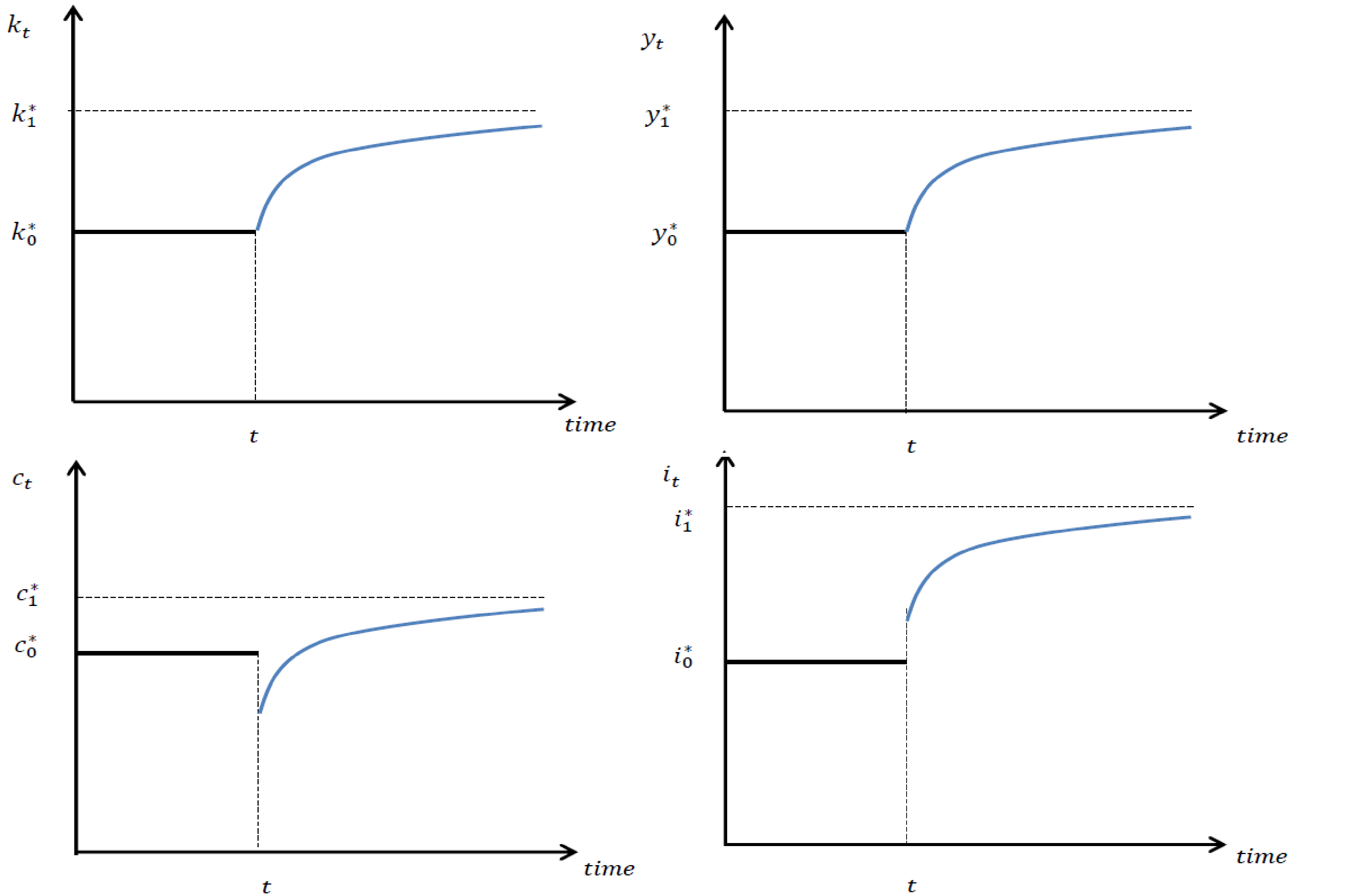
Note now that higher saving has offsetting effects on steady state consumption. We will return to that later.

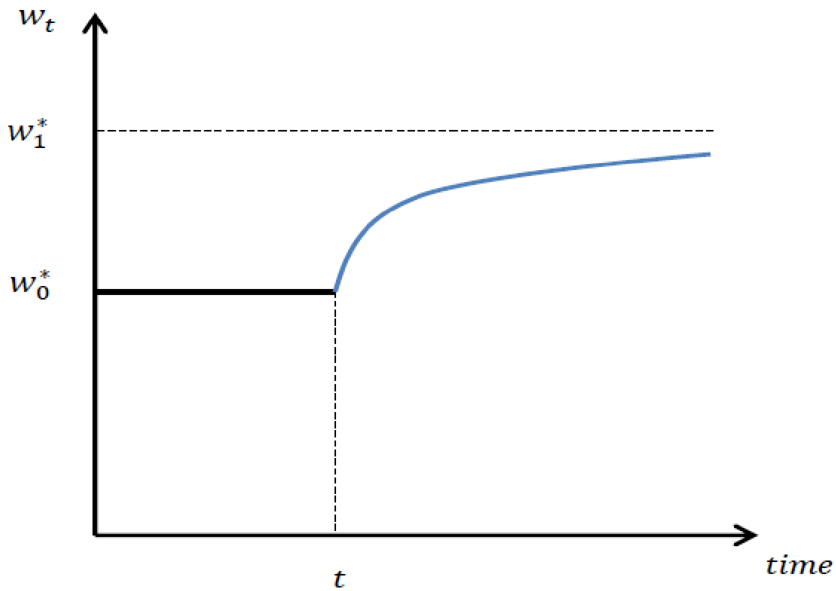
Dynamic effects of an increase in s

We will consider graphically the dynamic effects of an increase in the saving rate.

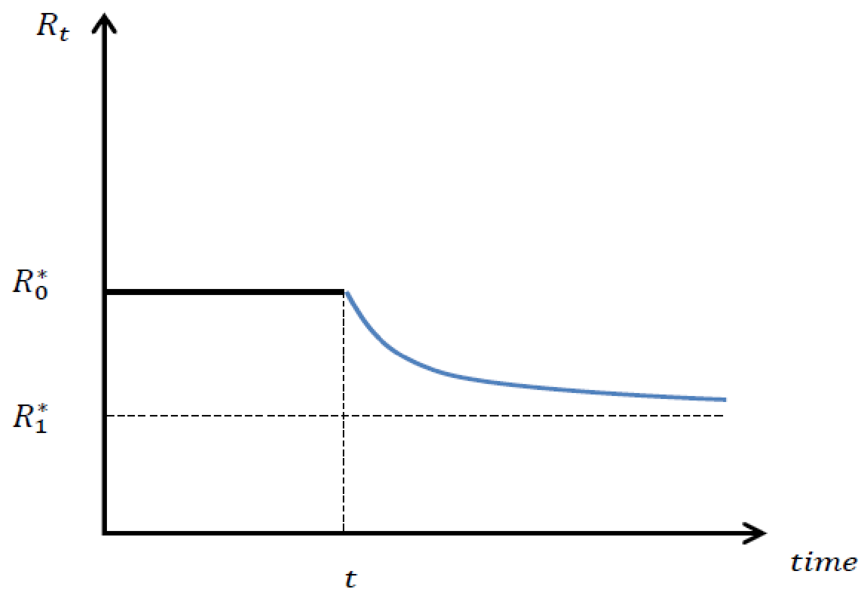


k_{t+1}
$k_{t+1} = k_t$
$k_{t+1} = s_1 A f(k_t) + (1-\delta)k_t$
$k_{t+1} = s_0 A f(k_t) + (1-\delta)k_t$
k_1^*
k_{t+2}
k_{t+1}
$k_t = k_0^*$
k_{t+1}
k_1^*
k_t

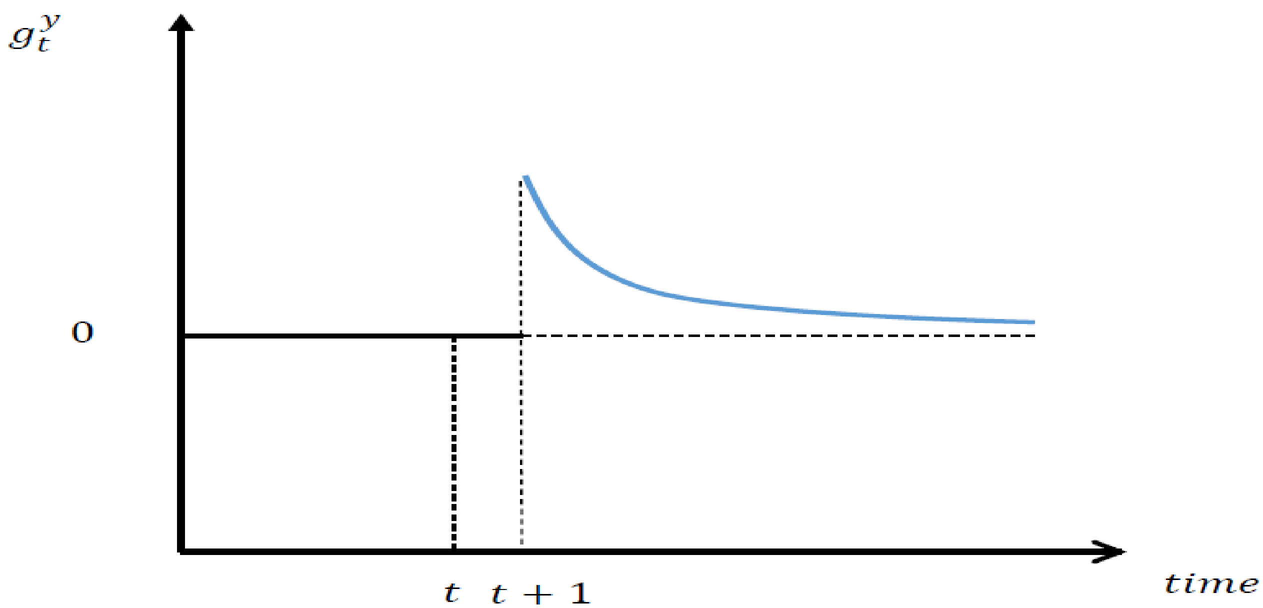

k_t
k_1^*
k_0^*
t
$time$
y_t
y_1^*
y_0^*
t
$time$
c_t
c_1^*
c_0^*
t
$time$
i_t
i_1^*
i_0^*
t
$time$



w_t
w_1^*
w_0^*
t
$time$



R_t
R_0^*
R_1^*
t
$time$

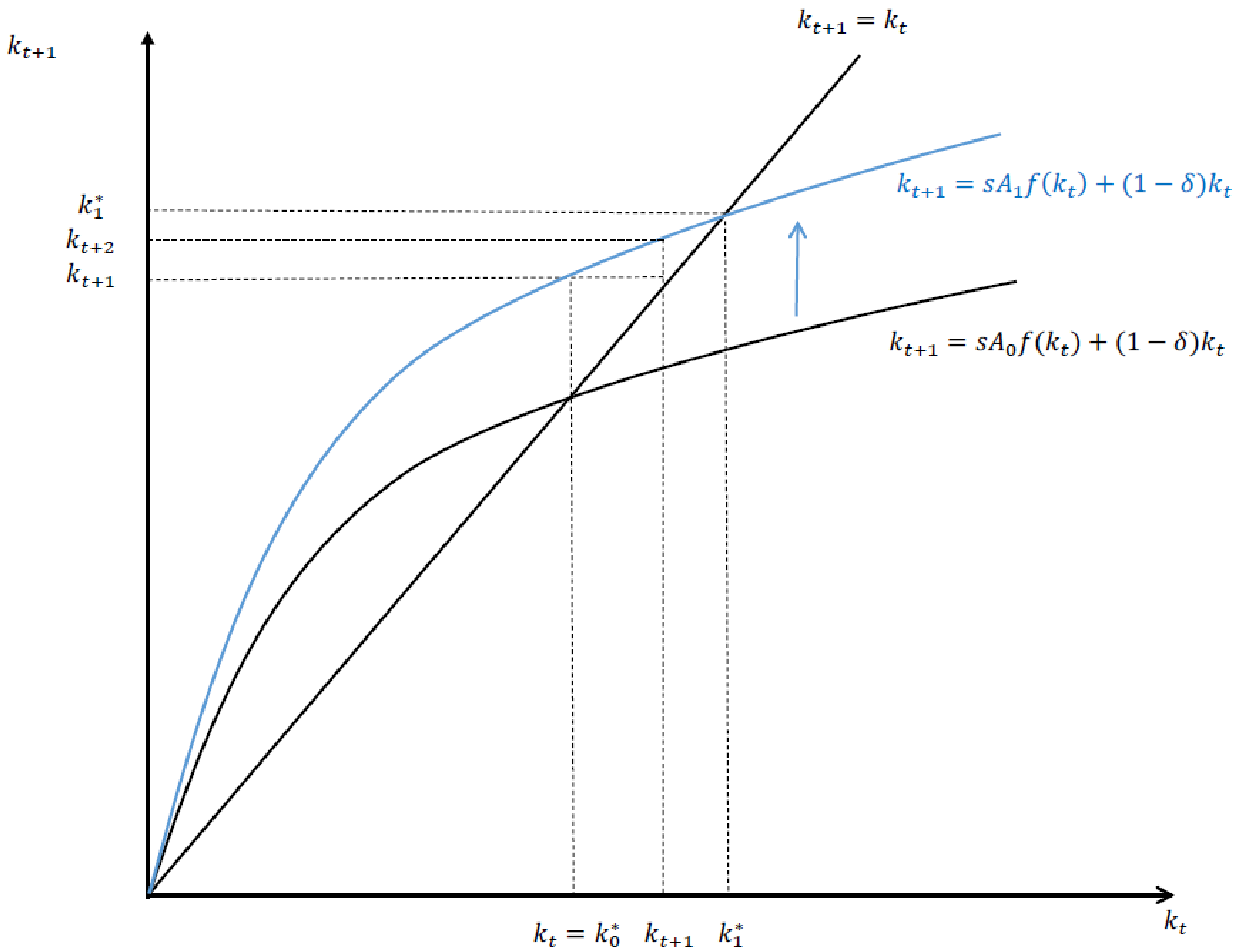


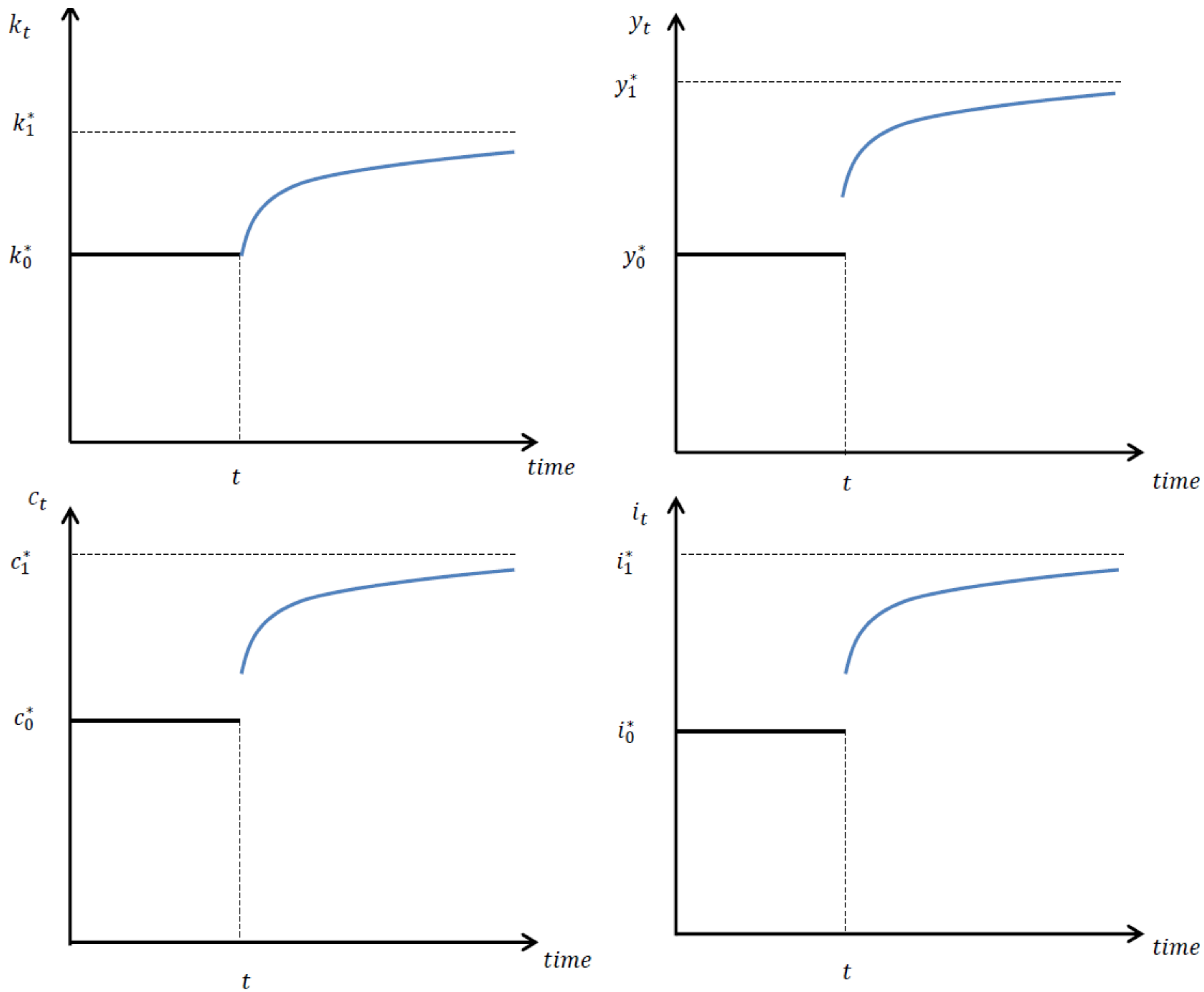
g_t^y
0
t
$t+1$
$time$

Next, the dynamic effects of a one-time increase in *A*

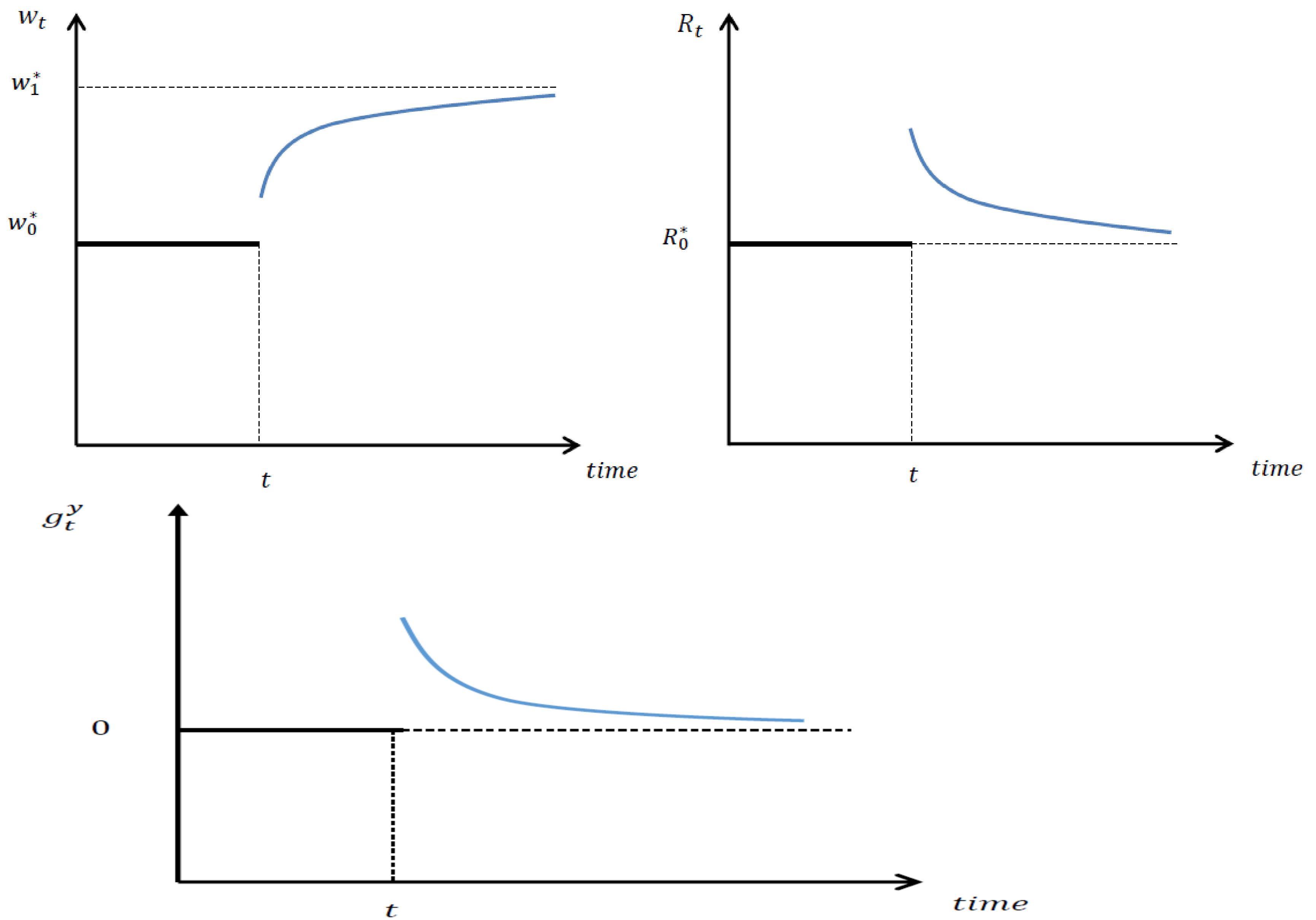
What leads to higher aggregate productivity?

- Each firm is more productive?
- New inventions?
- Better trained workers?
- Better allocation of resources across firms
  - Which may arise because of better financial markets
- Better government or better laws?
- Geographic forces?


k_{t+1}
$k_{t+1} = k_t$
$k_{t+1} = sA_1 f(k_t) + (1 - \delta)k_t$
k_1^*
k_{t+2}
k_{t+1}
$k_{t+1} = sA_0 f(k_t) + (1 - \delta)k_t$
$k_t = k_0^*$
k_{t+1}
k_1^*
k_t



k_t
k_1^*
k_0^*
t
time
y_t
y_1^*
y_0^*
t
time
c_t
c_1^*
c_0^*
t
time
i_t
i_1^*
i_0^*
t
time


w_t
w_1^*
w_0^*
t
$time$
R_t
R_0^*
t
$time$
g_t^y
O
t
$time$

Assessment

An increase in the saving rate or an increase in productivity lead to higher output in the long run.

They do not lead to higher growth in the long run.

However, while the saving rate cannot grow forever, perhaps productivity can. That may be a source for long run growth.

There is certainly a relationship between TFP and income per worker in the data.

The relationship between the saving rate and incomer per worker is weaker (but measures usually do not include saving by accumulating human capital.)

Cobb-Douglas case

The CD production function is not just a simple example. It helps us understand some of the "facts".

It was noted that the return to capital, R, is approximately constant over time. Also, the capital/output ratio $\frac{K}{Y} = \frac{k}{y}$ is roughly constant.

Together those imply that capital's share, $\frac{RK}{Y} = \frac{Rk}{y}$ is constant.

The facts also noted that labor's share, $\frac{wN}{Y} = \frac{w}{y}$ is roughly constant.

These are all properties of the CD production function.

Recall, $Y = AK^{\alpha} N^{1-\alpha}$.

$$R = \frac{\partial Y}{\partial K} = \alpha AK^{\alpha-1} N^{1-\alpha} = \alpha \frac{Y}{K}, \text{ which implies } \frac{RK}{Y} = \alpha .$$

Capital's share is a constant, α. Since there are no profits under constant-returns-to-scale, labor's share is simply $1-\alpha$.

We saw previously that in steady state, $k^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}}$ and $y^* = Ak^{\alpha}$.

Hence, $\frac{k^*}{y^*} = \left(\frac{sA}{\delta}\right) \Big/ A = \frac{s}{\delta}$. Productivity growth does not affect capital/output ratio.

$R = \alpha y^* / k^* = \alpha\delta / s$, which also does not depend on A.

Consumption

Since saving increases long-run output, should we save as much as possible?

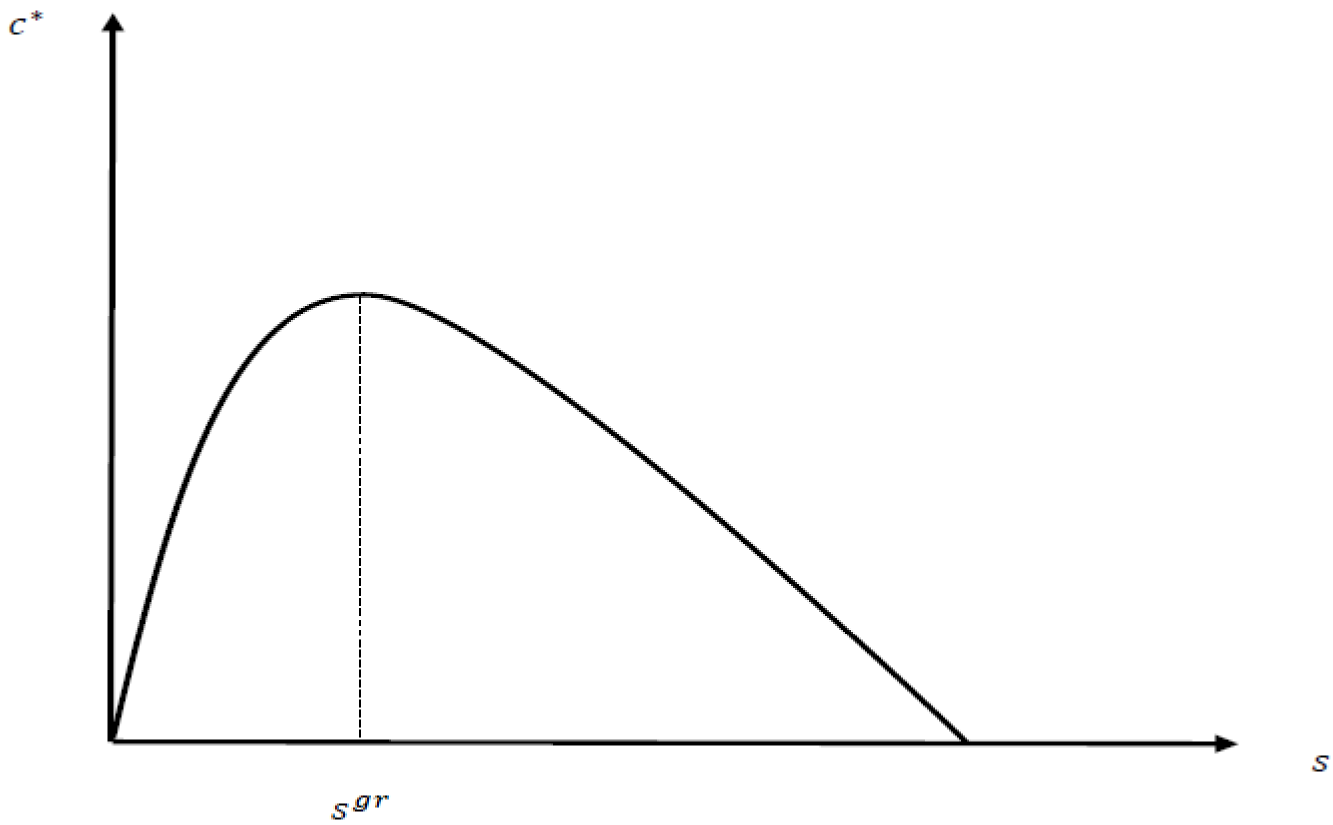
Ultimately, the objective of output is to allow for consumption. If we save too much, we are not leaving enough output for consumption.

So there is a tradeoff.

What level of saving maximizes long-run consumption?

We have $c^* = (1-s)Af\left(k^*\right)$, where k^* satisfies $sAf\left(k*\right) = \delta k^*$.

To find the optimal level of s, we want to choose s to maxmize c^*.


c^*
s
s^{gr}

$$c^* = (1-s)Af(k^*)$$

First-order condition: $\dfrac{dc^*}{ds} = -Af(k^*) + (1-s)Af'(k^*)\dfrac{dk^*}{ds} = 0$

We need to find $\dfrac{dk^*}{ds}$. We had $sAf(k*) = \delta k^*$. Then:

$$Af(k*) + sAf'(k*)\frac{dk^*}{ds} = \delta\frac{dk^*}{ds}$$, which we solve to find

$$\frac{Af(k*)}{\delta - sAf'(k*)} = \frac{dk^*}{ds}.$$

Subsitute this into the first-order condition:

$$\frac{dc^*}{ds} = -Af\left(k^*\right) + (1-s)Af'\left(k^*\right)\left[\frac{Af(k*)}{\delta - sAf'(k*)}\right] = 0$$

Now simplify this expression:

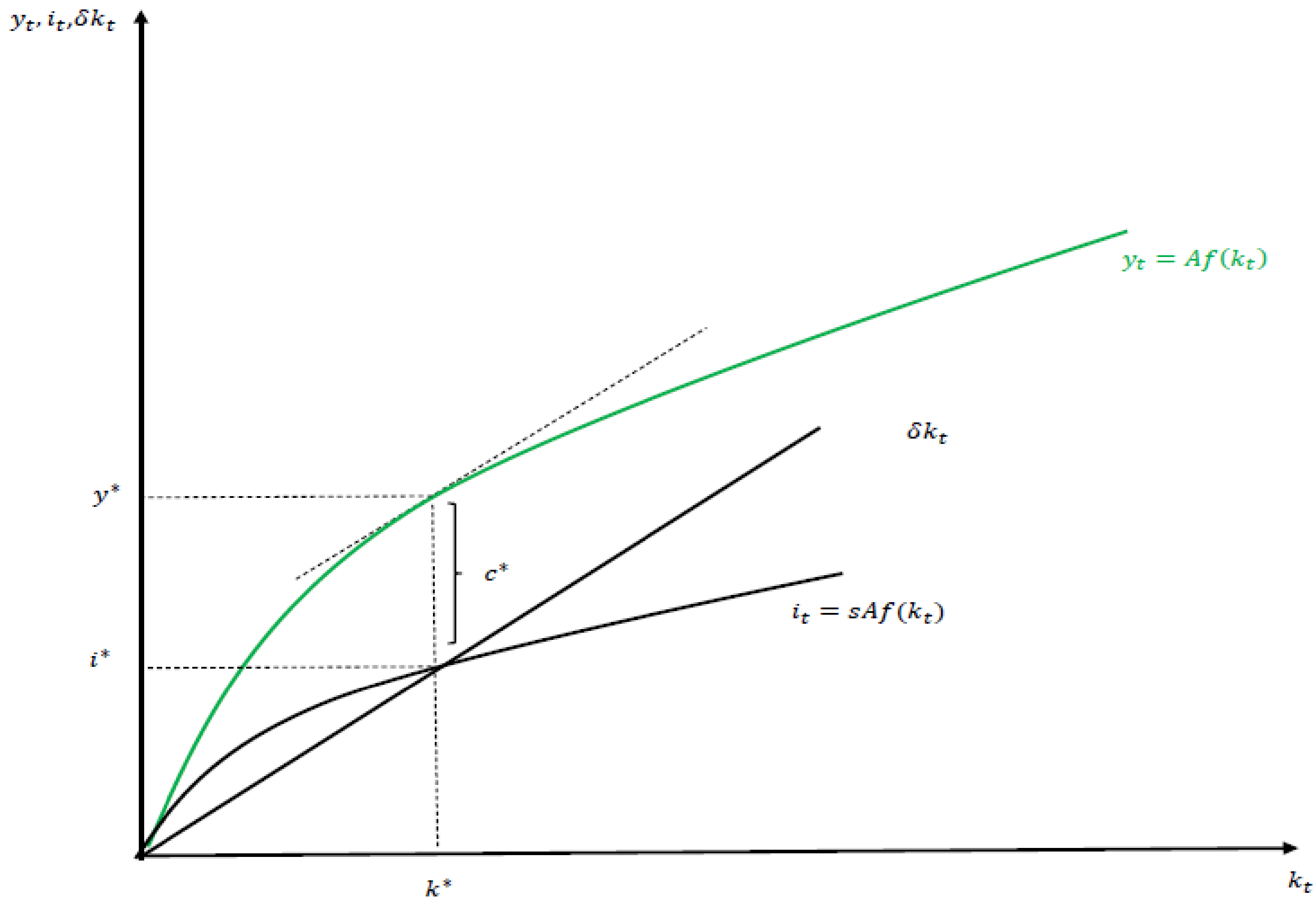
$$-1 + (1-s)Af'\left(k^*\right)\left[\frac{1}{\delta - sAf'(k*)}\right] = 0$$

$$(1-s)Af'\left(k^*\right)\left[\frac{1}{\delta - sAf'(k*)}\right] = 1$$

$$(1-s)Af'\left(k^*\right) = \delta - sAf'(k*)$$

$$Af'\left(k^*\right) = \delta$$

That is, we want the saving rate that gives us $Af'\left(k^*\right) = \delta$.

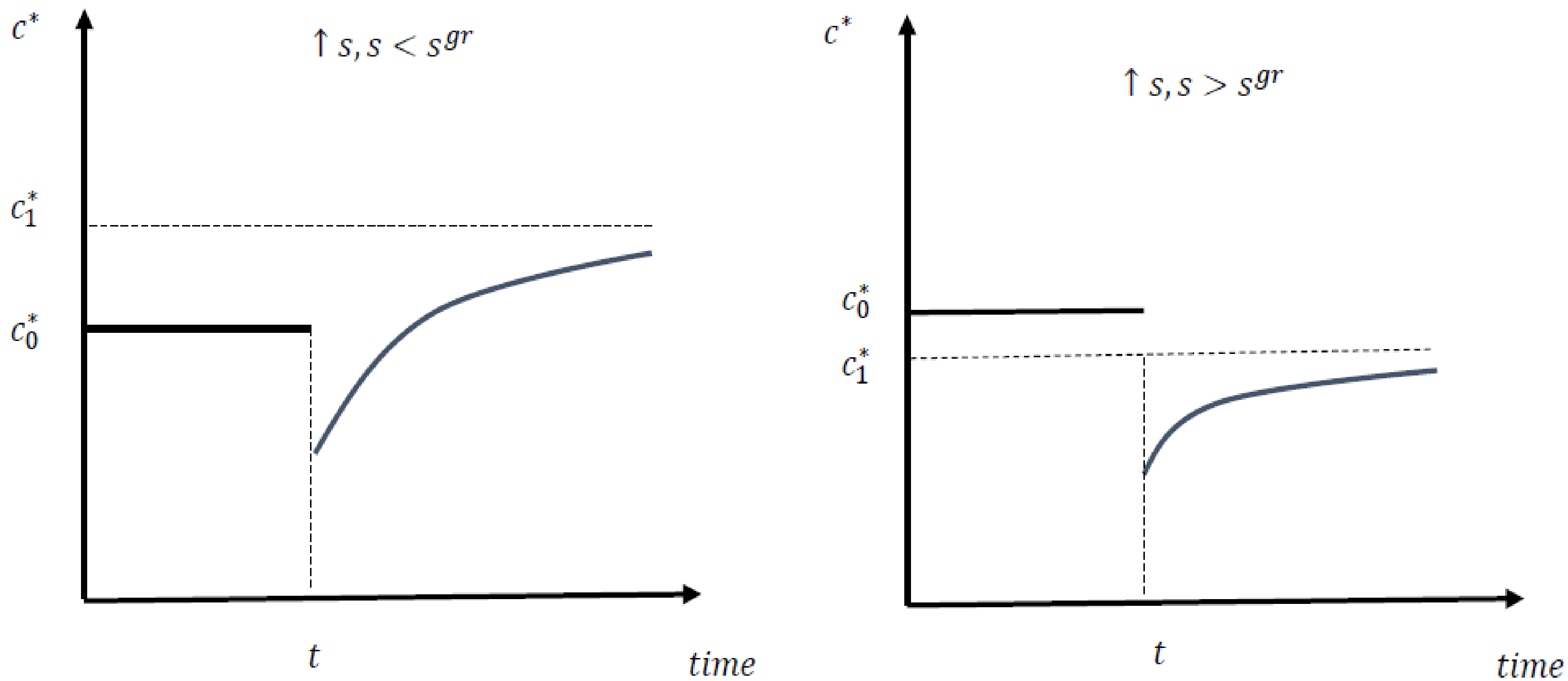

$y_t, i_t, \delta k_t$
$y_t = Af(k_t)$
δk_t
y^*
c^*
$i_t = sAf(k_t)$
i^*
k^*
k_t

Intuition:

Suppose $Af'\left(k^*\right) > \delta$. Then raising the capital stock raises steady-state output by $Af'\left(k^*\right)$, and raises steady-state investment by δ. If output goes up more than investment, than consumption will rise.

The reverse is true if $Af'\left(k^*\right) < \delta$.

Call the level of saving that maximizes steady-state consumption the "golden rule" level of saving, s^{gr}. Now examine how consumption evolves over time if the saving rate rises:



In the second case, when the saving rate is already above the golden rule level, and increase in saving lowers consumption in all periods!

This is "dynamically inefficient".

Is it optimal to maximize steady-state consumption?

In a sense, we spend almost all our time "near" the steady state. You can define "near" arbitrarily – say we are near the steady state if $\left|k_t - k^*\right| < \varepsilon$ for some arbitrary ε. For any ε, the fraction of time we spend near the steady state is equal to one.

So, if we give equal weight to the future as to the present, then we do want to maximize steady-state consumption per person.

But as we shall see later, people probably put more weight on current consumption than future consumption. Since we must give up current consumption to get future consumption, it will not in general be optimal to maximize steady-state consumption.

Optimal saving will be even lower than s^{gr}!

Lucas Paper

Lucas presents a version of the Solow growth model that is similar to the one we will look at in the next chapter. However, the consumption/saving decision is optimal (rather than the assumption that a fixed fraction of income is saved.)

Lucas emphasizes the fact that higher saving, or a higher level of productivity, have level effects on output in the long run, but do not have growth rate effects in the long run.

If we have $y_t = A_t k_t^{\alpha}$, then it follows that $g_t^y = g_t^A + \alpha g_t^k$. If the capital/labor ratio reaches a steady state, output per worker can only grow if TFP grows. (In the next chapter, we introduce "labor enhancing productivity." In that model, TFP is constant, but we get long-run growth because *k* grows in the long run.)

Lucas points out that if the Solow model is right, then low-income countries must have very low capital/labor ratios. But the Solow model assumes no interaction with the rest of the world. Why doesn't capital flow from the rest of the world?

Human Capital

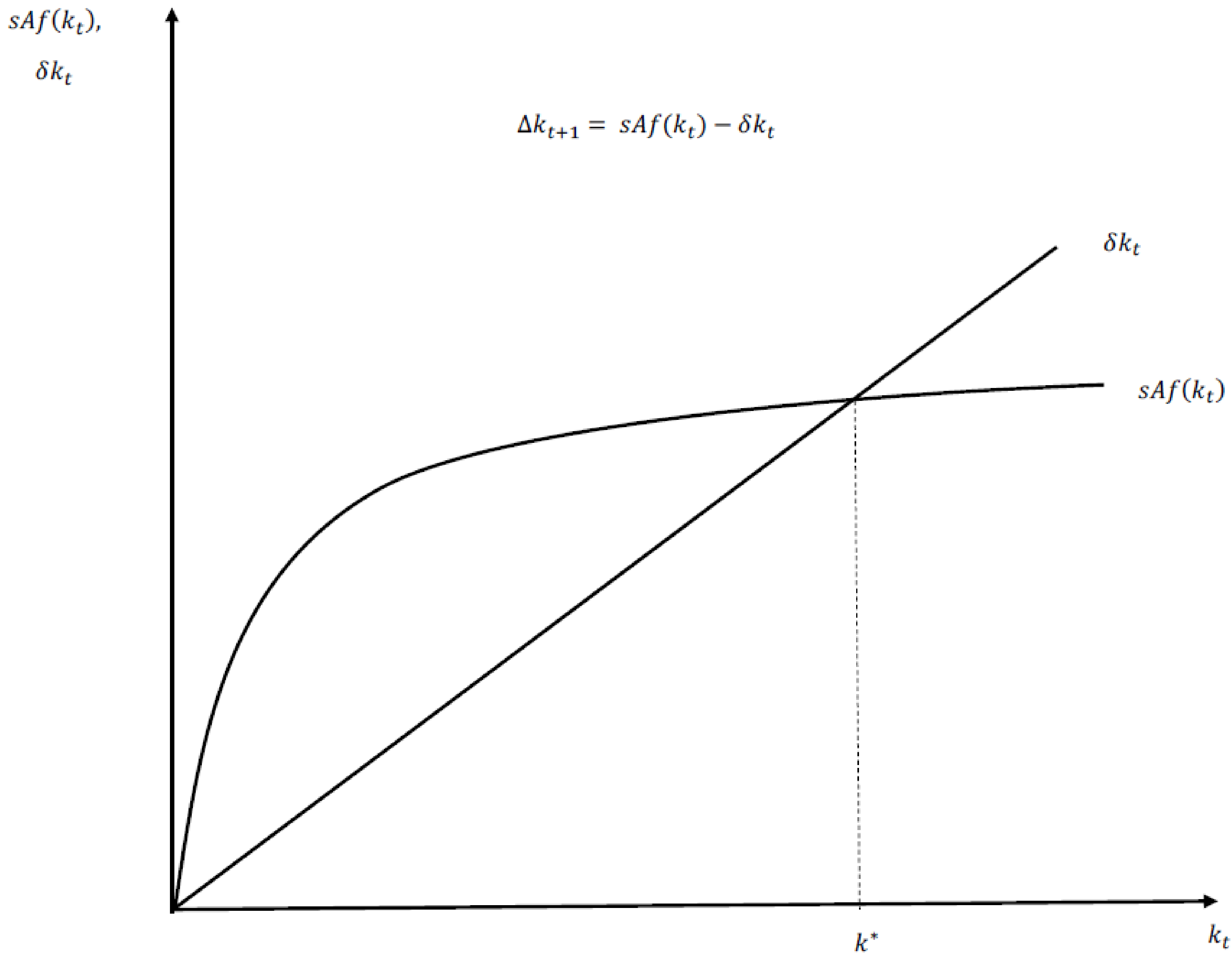
Lucas also emphasizes the importance of human capital.

Suppose we interpreted K as human capital in this model.

Human capital does not depreciate. That is, we might have $\delta = 0$.

How does the Solow model look when $\delta = 0$?

Recall this graph:


$sAf(k_t)$,
δk_t
$\Delta k_{t+1} = sAf(k_t) - \delta k_t$
δk_t
$sAf(k_t)$
k^*
k_t

If $\delta = 0$, growth would continue forever.

We have in that case simply $k_{t+1} - k_t = sAf'(k_t)$.

But there is something unreasonable about assuming a constant saving rate.

Under the Inada conditions, as $k_t \to \infty$, $f'(k_t) \to 0$.

It would be weird to keep saving a constant fraction of income (and investing in human capital), even as the rate of return on human capital goes to zero.

We will look at models of optimal consumption/saving soon. We might derive an equation like this:

$\frac{C_{t+1}}{C_t} = \beta\left(1+R_{t+1}\right)$, where β is positive but less than one. It means that people "discount" future consumption relative to current consumption.

Otherwise, let's take the rest of the Solow model as before, with the Cobb-Douglas production function, but with $\delta = 0$. To keep things simple, let us set $N_t = 1$.

Then $R_t = \alpha A K_t^{\alpha-1}$, and

$$K_{t+1} - K_t = A_t K_t^{\alpha} - C_t$$

We can express the model in two equations:

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K_{t+1}^{\alpha-1}\right) \qquad \text{and } K_{t+1} - K_t = A_t K_t^{\alpha} - C_t$$

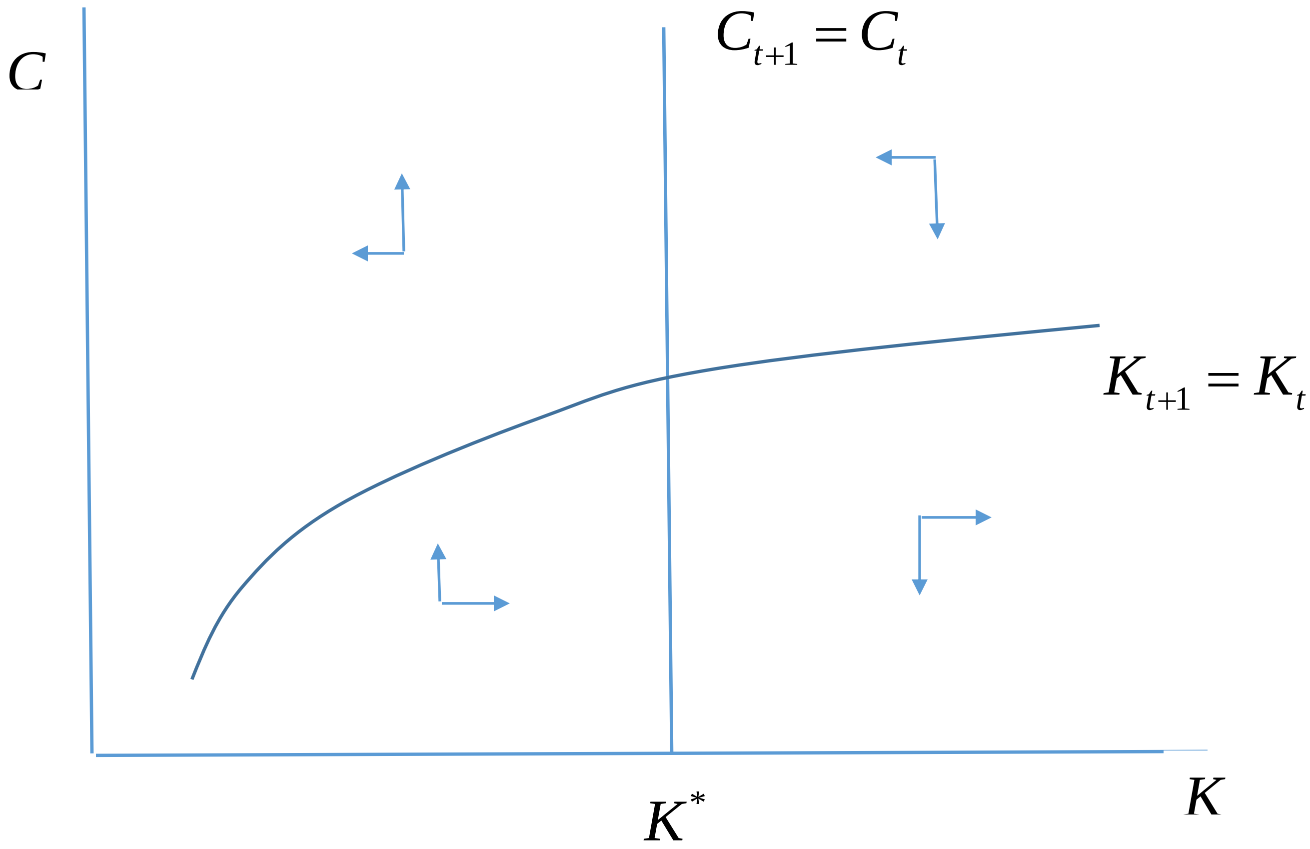
What is the steady state? We have $\frac{C_{t+1}}{C_t} = 1$, so $1 = \beta\left(1 + \alpha A\left(K^*\right)^{\alpha-1}\right)$.

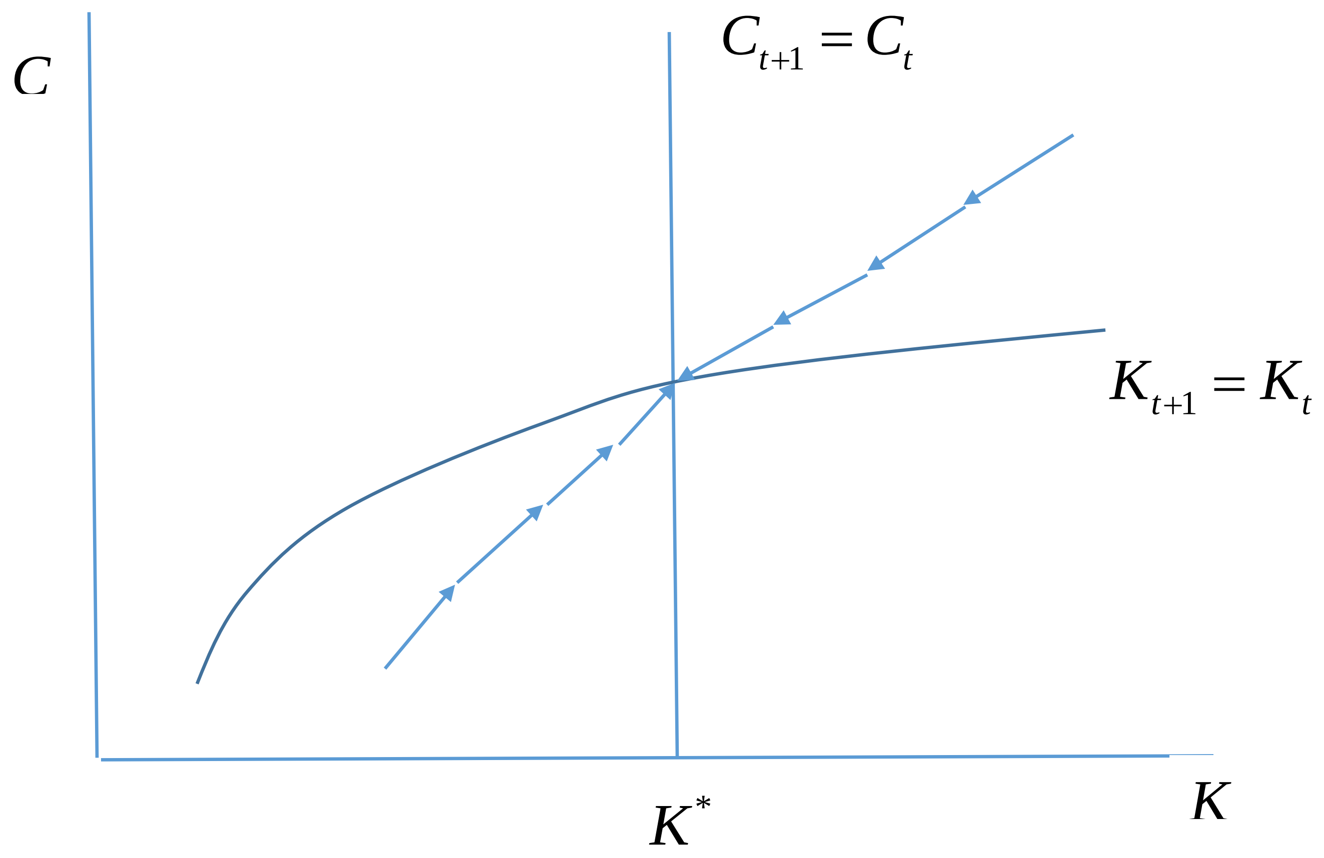
This tells us that $K^* = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$.

In the steady state, $C^* = Y^* = AK^{*\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$.

Higher productivity (A), greater "patience" (higher β), and a greater capital share (α), lead to higher consumption and income in the long run.

We can look at dynamics in a "phase diagram"




C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K^*
K

Lesson from this model:

We still have a steady state with no long run growth!

Saving more this period, increases the capital stock next period.

As the capital stock increases, the marginal product of capital falls.

Output approaches a steady state.

The optimal consumption path when below the steady state is to have high consumption growth initially. But as the MPK falls, consumption growth falls, and consumption growth approaches zero as R^* goes toward $\frac{1-\beta}{\beta}$.