Econ 702

Macroeconomics I

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Spring 2020

Lecture 4: Augmented Solow Growth Model

We will now turn to a version of the Solow growth model where there is long-run growth.

As our previous discussion has suggested, the key to long-run growth will be long-run productivity growth.

You might think the way to introduce this into the model is to have TFP, A_t, grow over time.

We could do that, but it would give us the result that returns to capital grow over long periods of time. Our facts told us that was not true – real wages grow over time but returns to capital are pretty constant.

Instead, we will introduce “labor augmenting productivity”, Z_t.

$$Y_t = AF(K_t, Z_t N_t)$$

Notice how Z_t enters the production function. An increase in Z_t works exactly like increasing the number of workers. In fact, we can think of $Z_t N_t$ as being the “effective” number of workers, or the “efficiency units” of labor.

One difference between Z_t and A is that there is diminishing marginal product of Z_t. Holding factor inputs constant, increasing Z_t increases output, but at a diminishing rate. But an increase in A always leads to a proportionate increase in output.

We will let Z_t grow over time, and also the workforce, N_t. Otherwise we don't change the Solow model. And through some clever algebra, the analysis of the model is very similar.

We have:

$$K_{t+1} = I_t + (1-\delta)K_t$$

$$I_t = sY_t.$$

$$R_t = AF_K(K_t, Z_t N_t)$$

$$w_t = AZ_t F_N(K_t, Z_t N_t)$$

Notice that Z_t affects the marginal product of labor not only by increasing the effective number of workers, but also directly by increasing each one's productivity.

We will assume the work force grows at a rate *n*, and that the initial work force (in time 0) is equal to one:

$$N_t = (1+n)N_{t-1}$$
$$N_0 = 1$$
$$N_t = (1+n)^t$$

Similarly, we will assume labor augmenting productivity grows at a rate *z*, and in time 0 is equal to one:

$$Z_t = (1+z)Z_{t-1}$$
$$Z_0 = 1$$
$$Z_t = (1+z)^t$$

The equations of the model are therefore given by:

$$
\begin{aligned}
K_{t+1} &= I_t + (1-\delta)K_t \\
I_t &= sY_t \\
Y_t &= AF(K_t, Z_t N_t) \\
Y_t &= C_t + I_t \\
R_t &= AF_K(K_t, Z_t N_t) \\
N_t &= (1+n)^t \\
Z_t &= (1+z)^t \\
w_t &= AZ_t F_N(K_t, Z_t N_t)
\end{aligned}
$$

As before, we can summarize the dynamic evolution of the capital stock in the net investment equation:

$$K_{t+1} = sAF(K_t, Z_t N_t) + (1 - \delta)K_t$$

Previously, we transformed the model by expressing the key variables in "per worker" terms (dividing by N_t.)

Now we will transform the model by expressing the key variables in "per efficiency unit of labor terms" (dividing by $Z_t N_t$.)

For some variable X_t (such as K_t, Y_t, C_t, etc.) we define the per efficiency unit of labor variable $\hat{x}_t$ as

$$\hat{x}_t = \frac{X_t}{Z_t N_t}$$

Follow similar steps as before:

$$\frac{K_{t+1}}{Z_t N_t} = \frac{sAF(K_t, Z_t N_t)}{Z_t N_t} + (1-\delta)\frac{K_t}{Z_t N_t}$$

Then we use the constant-returns-to-scale assumption:

$$\frac{F(K_t, Z_t N_t)}{Z_t N_t} = F\left(\frac{K_t}{Z_t N_t}, \frac{Z_t N_t}{Z_t N_t}\right) = F(\widehat{k}_t, 1)$$

We get

$$\frac{K_{t+1}}{Z_t N_t} = sAf(\widehat{k}_t) + (1-\delta)\widehat{k}_t$$

where we define

$$f(\widehat{k}_t) = F(\widehat{k}_t, 1)$$

Then multiply and divide the left-hand side by $Z_{t+1}N_{t+1}$:

$$\frac{K_{t+1}}{Z_{t+1}N_{t+1}}\frac{Z_{t+1}N_{t+1}}{Z_tN_t} = sAf(\widehat{k}_t) + (1-\delta)\widehat{k}_t$$

From our assumptions above, we have $\frac{Z_{t+1}N_{t+1}}{Z_tN_t} = (1+z)(1+n)$,

so we get:

$$\widehat{k}_{t+1} = \frac{1}{(1+z)(1+n)}\left[sAf(\widehat{k}_t) + (1-\delta)\widehat{k}_t\right]$$

This equation is very similar to the one we had previously for k_t. The only difference are the constant terms on the right side.

The other equations of the model (in efficiency units of labor terms) are:

$$\widehat{i_t} = s\widehat{y_t}$$

$$\widehat{y_t} = Af(\widehat{k_t})$$

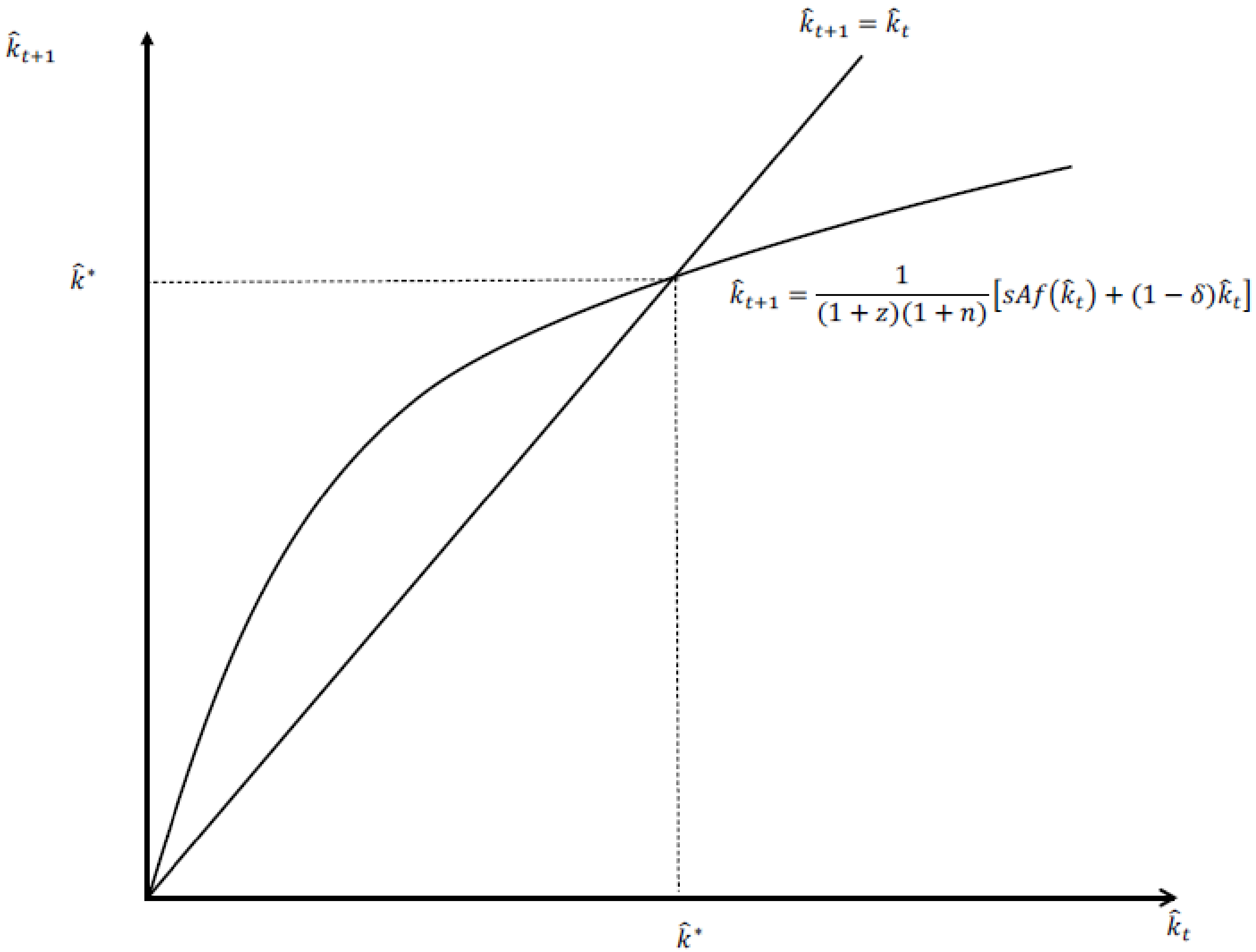
$$\widehat{y_t} = \widehat{c_t} + \widehat{i_t}$$

$$R_t = Af'(\widehat{k_t})$$

$$w_t = Z_t\left[Af(\widehat{k_t}) - Af'(\widehat{k_t})\widehat{k_t}\right]$$

See the textbook for step-by-step derivation of these equations.

We can analyze the model graphically as before.



As before, we converge to a steady state.

Analysis of Steady State

In the steady state, $\hat{k}_{t+1} = \hat{k}_t$, which means $\frac{K_{t+1}}{Z_{t+1}N_{t+1}} = \frac{K_t}{Z_t N_t}$. That gives us:

$$\frac{K_{t+1}}{K_t} = \frac{Z_{t+1}N_{t+1}}{Z_t N_t}$$

$$\frac{K_{t+1}}{K_t} = (1+z)(1+n)$$

Let g_K be the growth rate of K_t in the steady state. We have:

$$1 + g_K = (1+z)(1+n)$$

$$\Rightarrow g_K \approx z + n.$$

In a lot of cases, we are more interested in the growth rate of per-worker or per-person variables. For capital per worker, k_t, we find:

$$\frac{K_{t+1}}{N_{t+1}} = \frac{Z_{t+1}}{Z_t}\frac{K_t}{N_t}$$

$$\frac{k_{t+1}}{k_t} = 1 + z \Rightarrow g_k = z$$

In other words, the capital stock grows at the sum of the growth rates of productivity and the work force. Capital per worker grows at the rate of growth of productivity.

For total output, consumption and investment, we find:

$$\frac{Y_{t+1}}{Y_t} = (1+z)(1+n) \Rightarrow g_Y \approx z+n$$
$$\frac{C_{t+1}}{C_t} = (1+z)(1+n) \Rightarrow g_C \approx z+n$$
$$\frac{I_{t+1}}{I_t} = (1+z)(1+n) \Rightarrow g_I \approx z+n.$$

For growth rates "per worker", we have:

$$\frac{y_{t+1}}{y_t} = (1+z) \Rightarrow g_y = z$$
$$\frac{c_{t+1}}{c_t} = (1+z) \Rightarrow g_c = z$$
$$\frac{i_{t+1}}{i_t} = (1+z) \Rightarrow g_i = z.$$

This is an important conclusion, because we find that output per worker and consumption per worker grow, even in the long run. The growth is driven by long-run labor-augmenting productivity growth.

Note, importantly, that the return to capital is constant in the steady state:

$$R^* = Af'(\widehat{k}^*)$$

But the real wage rate grows at the rate of labor-augmenting productivity growth:

$$w_t = Z_t\left[Af(\widehat{k}^*) - Af'(\widehat{k}^*)\widehat{k}^*\right]$$

Since $\hat{k}^*$ is constant,

$$\frac{w_{t+1}}{w_t} = \frac{Z_{t+1}}{Z_t}$$

$$= 1 + z$$

$$\Rightarrow g_w = z.$$

So, the model does match the key facts about the data (though the most important causal variable is exogenous!)

Let's briefly now turn to the dynamics of responses of variables (both per effecitive worker and per worker) to changes in *s* and *A*.

Figure 6.2: Increase in s

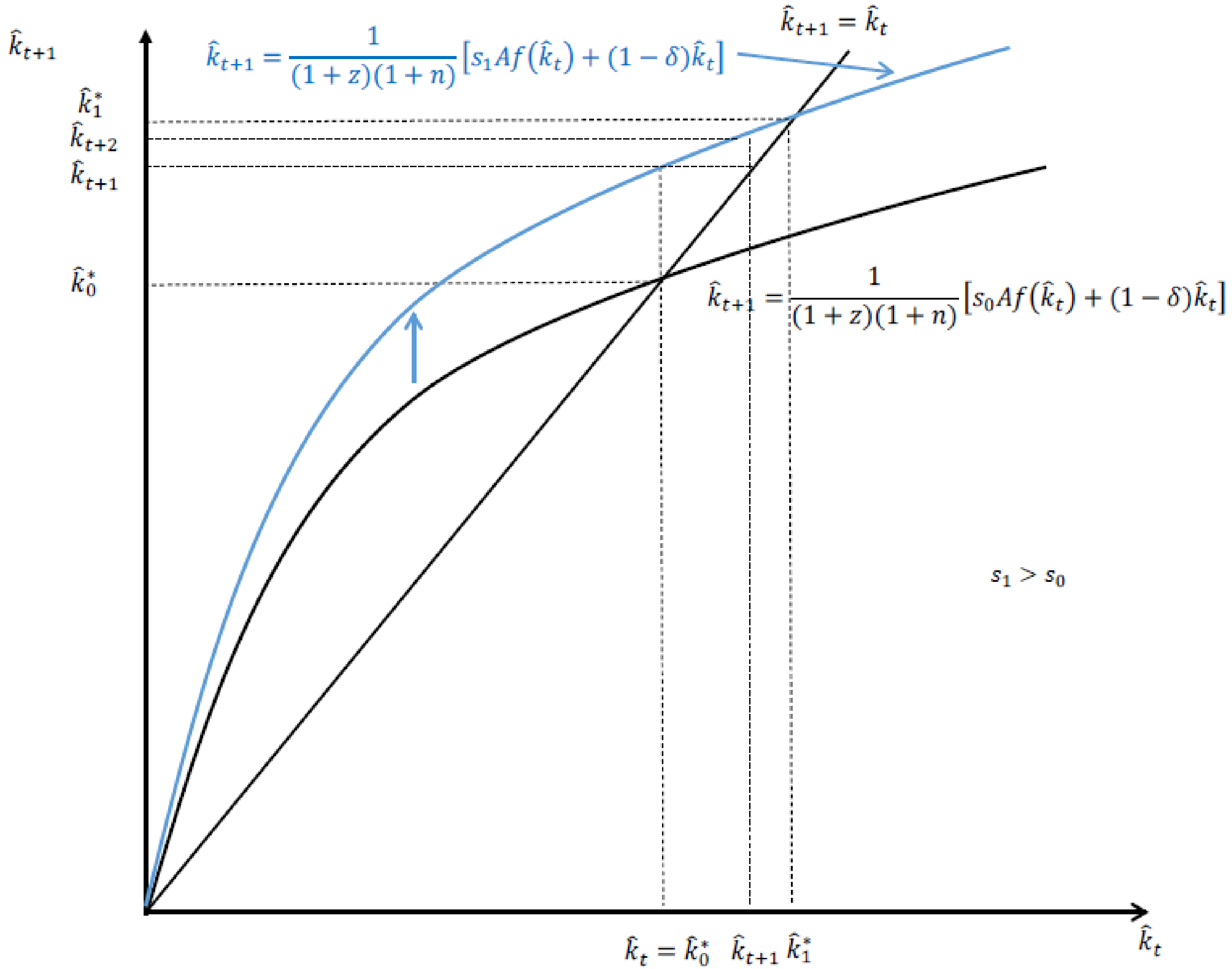


Figure 6.3: Dynamic Responses to Increase in s

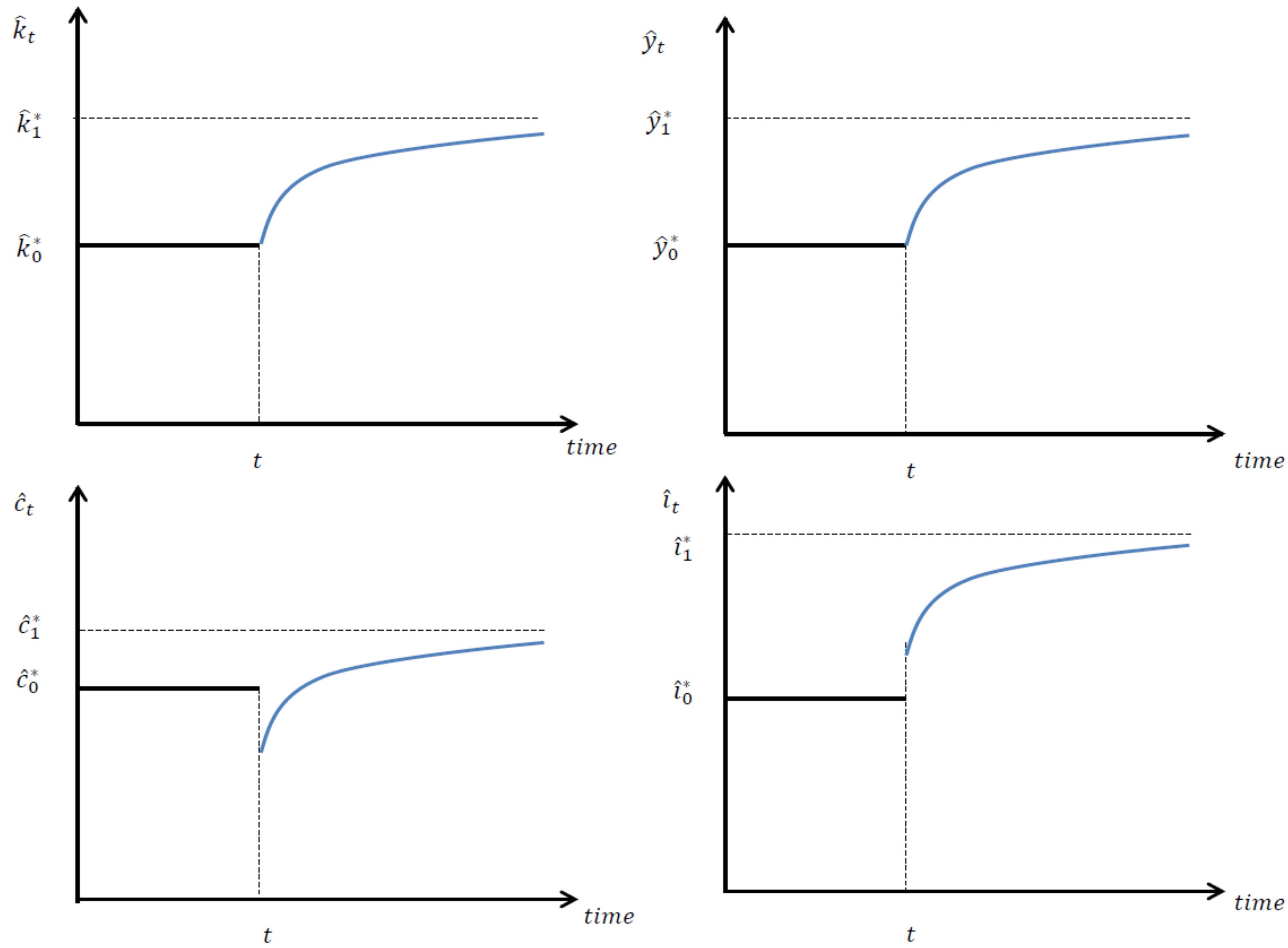
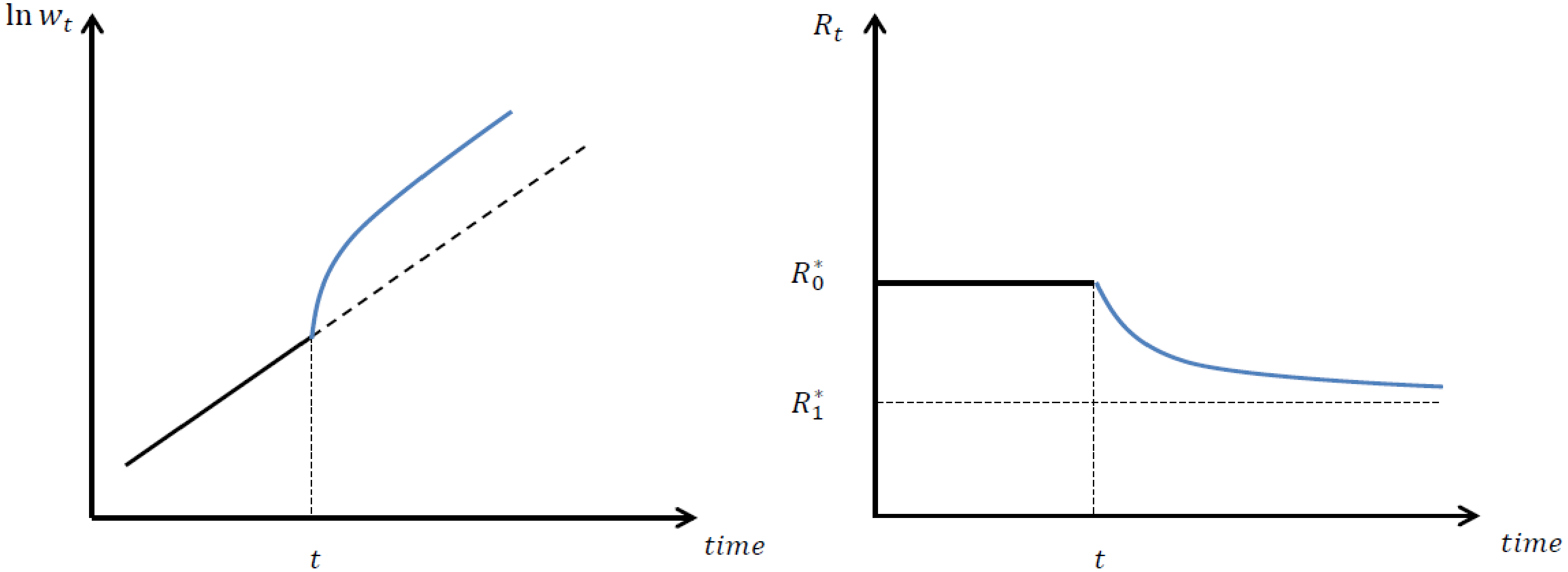


Figure 6.3: Dynamic Responses to Increase in s



We plot $\ln(w_t)$ because, approximately,

$$\ln(w_t) - \ln(w_{t-1}) \cong \frac{w_t - w_{t-1}}{w_{t-1}}$$

Figure 6.4: Dynamic Responses to Increase in s, Per Worker Variables

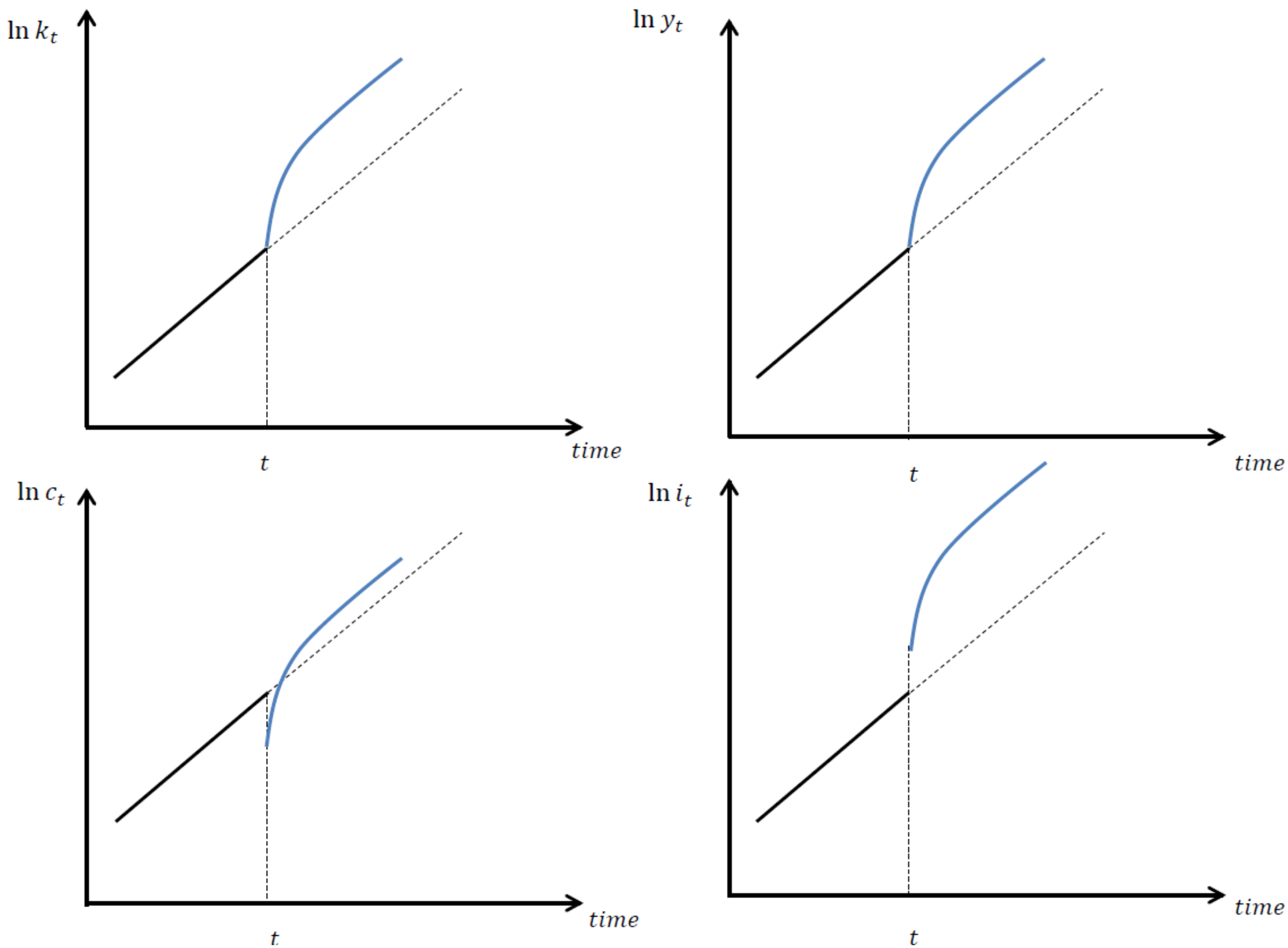


Figure 6.6: Increase in A

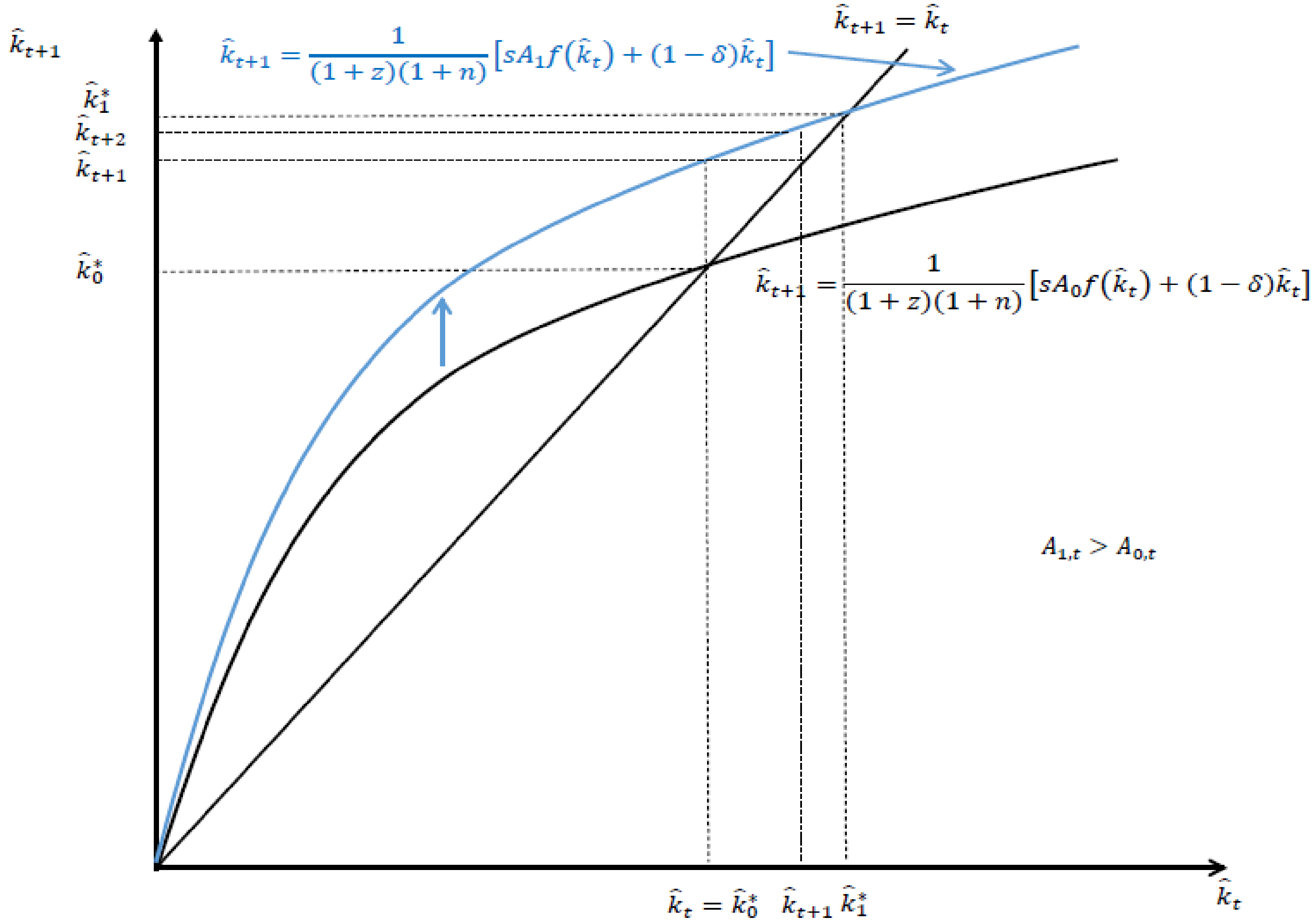


Figure 6.7: Dynamic Responses to Increase in A

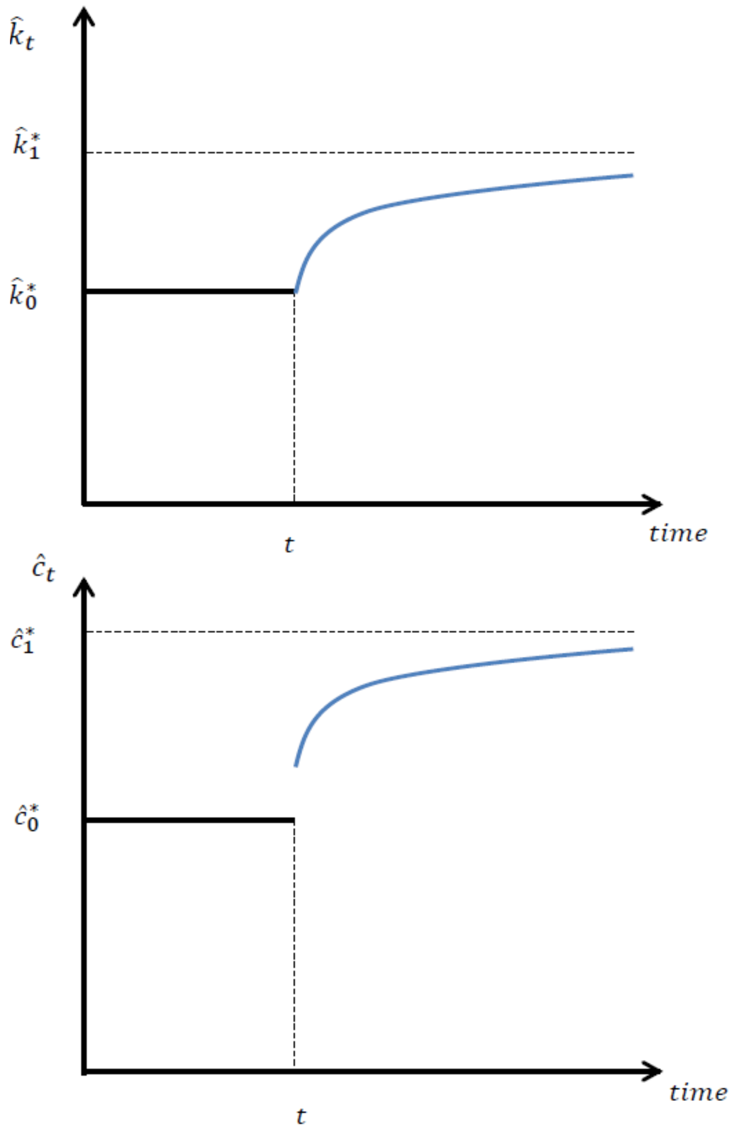


Figure 6.7: Dynamic Responses to Increase in A

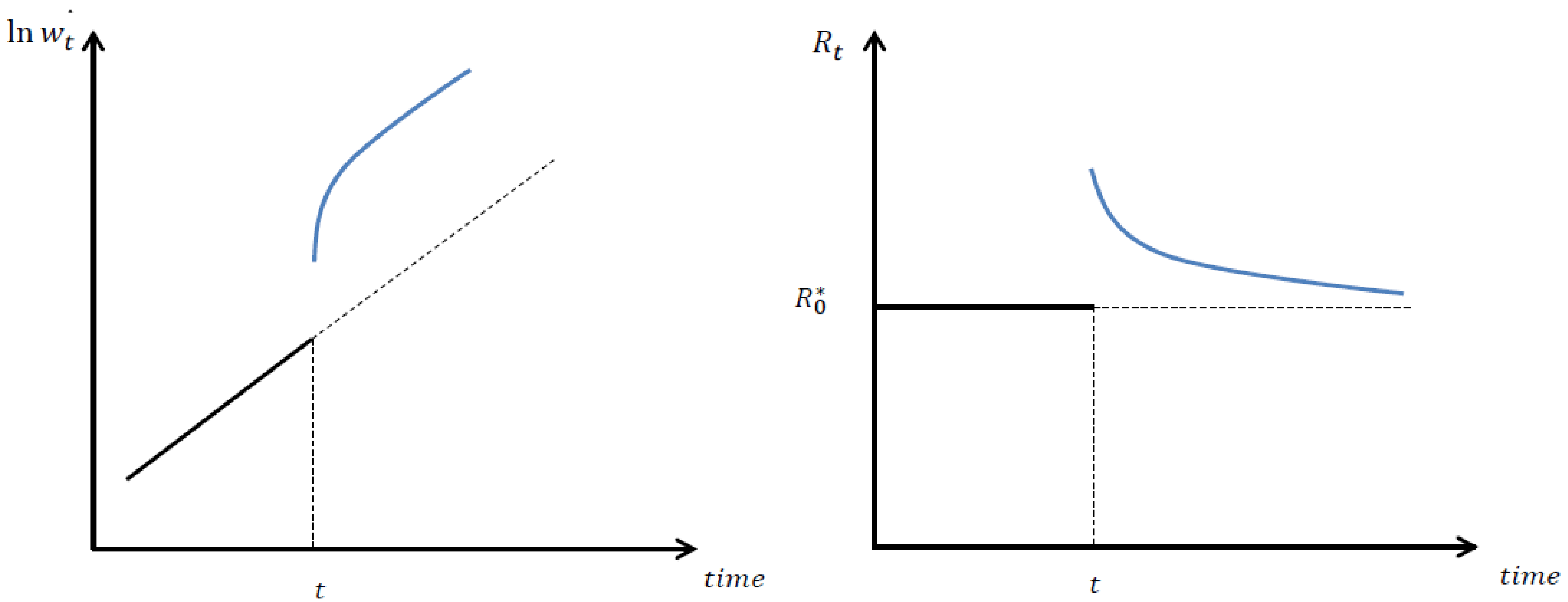
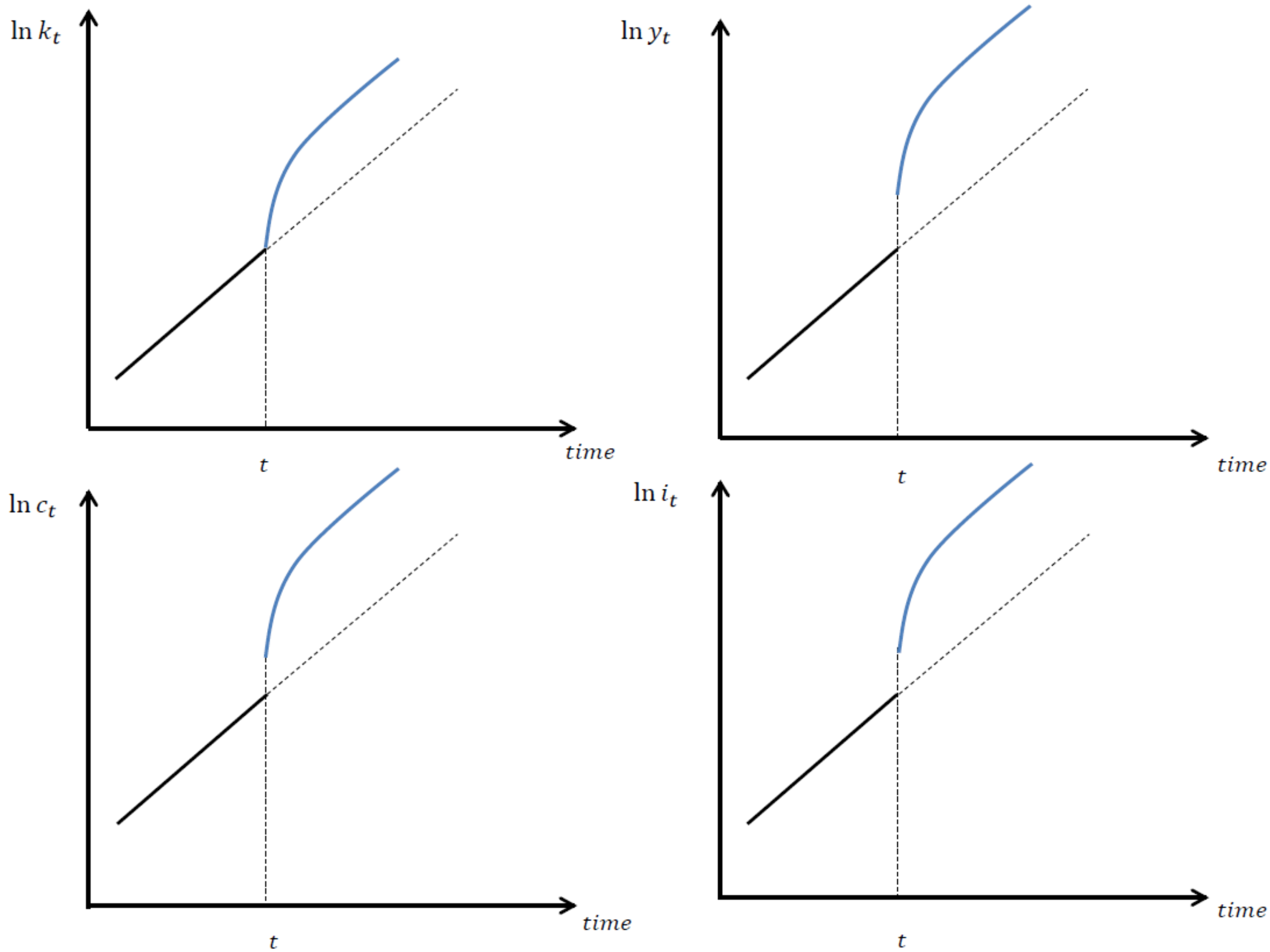


Figure 6.8: Dynamic Responses to Increase in A, Per Worker Variables



Conclusions

1. In order to have sustained growth in wages and consumption per person, we need sustained growth in productivity.
   a. In our model, productivity growth is exogenous.
   b. What might lead to greater productivity growth?

2. Higher saving rates or higher levels of productivity can lead to higher levels of income.
   a. If the adjustment to higher levels of income occurs gradually over long periods of time, then over those time periods, the growth rate will be higher.

Productivity growth

Productivity growth may occur through innovation.

What are the incentives to innovate?

"Nonrivalrous" – ideas are nonrivalrous. If one person uses an idea, that does not prevent someone else from using it.

If we think of "ideas" as factors of production, we recognize a difference between capital or labor, which are rivalrous, versus ideas which are nonrivalrous.

"Excludability" – a good (or idea) is excludable if it is possible to keep someone from using the good (or idea).

Incentives to Innovate

Innovation creates ideas. Ideas drive productivity growth.

If the idea is nonrivalrous, the inventor of the idea does not reap the full gains to creating the idea. Others get to use the idea but did not bear the cost of inventing the idea.

Policymakers can make ideas excludable, for example by issuing patents. People can use the idea only by paying the inventor of the idea.

There is a tradeoff from the point of view of the policymaker. Using the idea, once it is invented, is costless. So, there is underuse of the idea if people have to pay for it.

But if they do not pay for it, the inventor is underrewarded for his invention.

Some have proposed, for example, that the government buy patents, and then make inventions freely available. That way, the inventor gets rewarded, but the idea is used freely.

Even if an idea is patented, it might be imitated. Imitation is not free but it is cheaper than invention. Usually, within a country, an imitator must still pay the original inventor, unless the imitation is sufficiently different than the original invention.

To enforce a system of patents, a country needs a strong and impartial judicial system.

Note also the difficulty in enforcing patents in other countries.

There are treaties that enforce patents in other countries on inventions made in one country. However, the patents are not always enforced as strongly overseas. The imitating country has an incentive not to enforce the patent.

Enforcement of patents, and punishment for imitation are a subject for international trade negotiations.

We will discuss the role of “institutions” in leading to long-run growth.