Econ 702

Macroeconomics I

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Lecture 6: Optimal Consumption and Saving, part 1

The model of an optimizing household forms the basis for macroeconomic models of consumption and saving.

Dynamic

Two-period: Why?

Representative household

Periods t and $t+1$.

Household chooses C_t, S_t, C_{t+1}, S_{t+1} given Y_t, Y_{t+1}, r_t

Exogenous vs. endogenous

Household's budget constraints:

$$C_t + S_t \leq Y_t$$

$$C_{t+1} + S_{t+1} \leq Y_{t+1} + (1 + r_t)S_t$$

Thes latter can be written as:

$$C_{t+1} + S_{t+1} - S_t \leq Y_{t+1} + r_t S_t$$

("saving" and "savings")

Under optimality, we will have $S_{t+1} = 0$ and budget constraint will hold with equality.

We have then:

$$C_t + S_t = Y_t$$

$$C_{t+1} = Y_{t+1} + (1 + r_t) S_t$$

Solve out for S_t and rearrange to get a single constraint:

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t}$$

“present value” and “current value”

Households maximize

$$U = u(C_t) + \beta u(C_{t+1}), \quad 0 \leq \beta < 1$$

Marginal utility is positive, always:

$$u'(\cdot) > 0$$

Diminishing marginal utility:

$$u''(C_t) < 0$$

Some example utility functions:

$$u(C_t) = \theta C_t, \quad \theta > 0$$

$$u(C_t) = C_t - \frac{\theta}{2} C_t^2, \quad \theta > 0$$

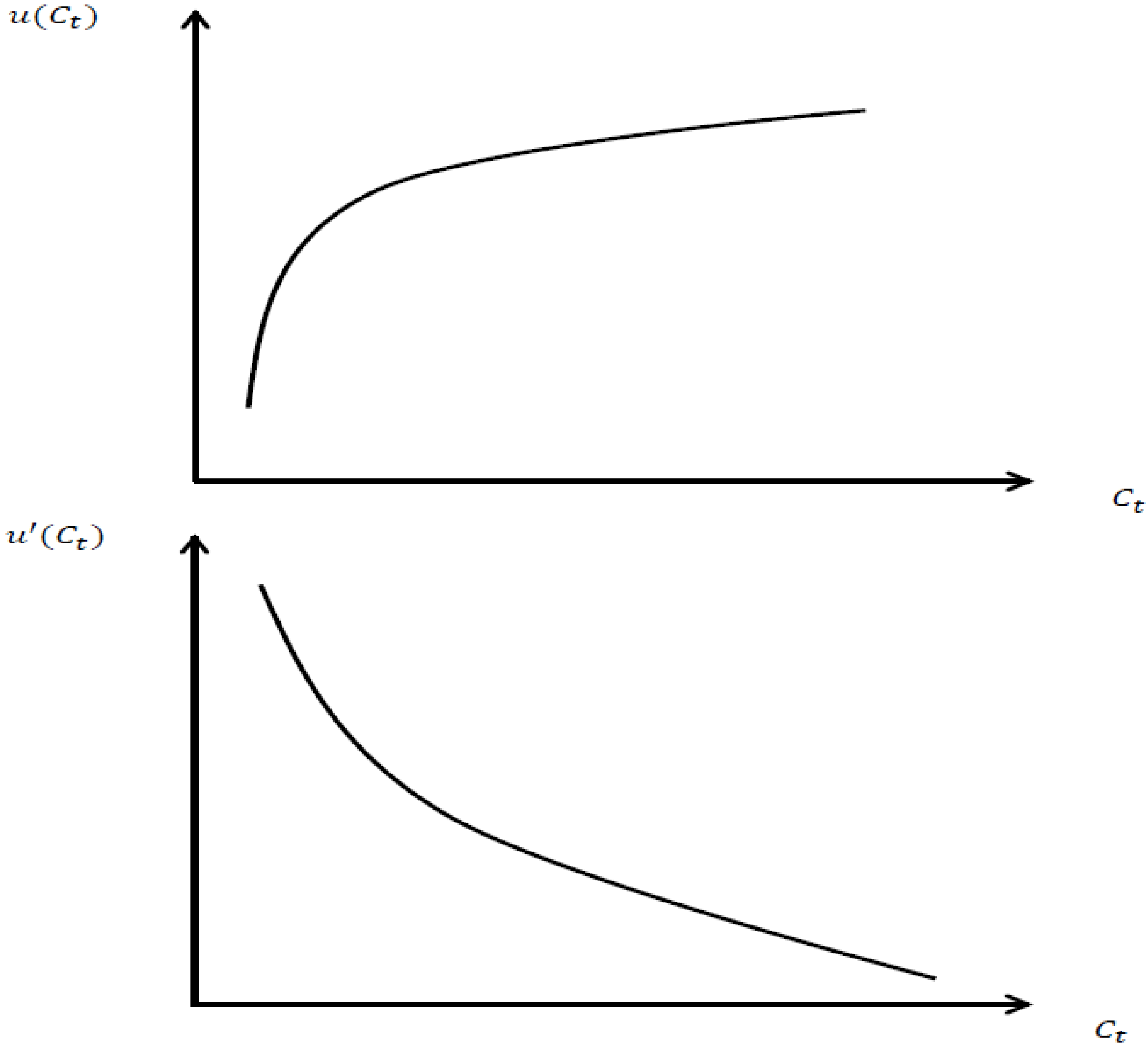
$$u(C_t) = \ln C_t$$

$$u(C_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{1}{1-\sigma}, \quad \sigma > 0$$

The first one is not concave (the second derivative is zero.)

The quadratic utility has the problem that the first derivative turns negative after a certain point.

Figure 9.1: Utility and Marginal Utility



The household's problem:

$$\max_{C_t, C_{t+1}} \quad U = u(C_t) + \beta u(C_{t+1})$$

subject to:

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t}.$$

We could use a Lagrangian to set up this problem. Instead, we will substitute out for C_{t+1}. That is, we will solve for C_{t+1} from the constraint:

$$C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1}$$

Then substitute this solution into the utility function to get an unconstrained maximization problem:

$$\max_{C_t} \quad U = u(C_t) + \beta u\left((1 + r_t)(Y_t - C_t) + Y_{t+1}\right)$$

The first-order condition is:

$$\frac{\partial U}{\partial C_t} = u'(C_t) - (1 + r_t)\beta u'\left((1 + r_t)(Y_t - C_t) + Y_{t+1}\right) = 0$$

But since $C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1}$, we can write

$$u'(C_t) - (1 + r_t)\beta u'(C_{t+1}) = 0$$

It is intuitive to write this as:

$$u'(C_t) = \beta(1 + r_t)u'(C_{t+1})$$

This is called the Euler equation.

We could write this as the marginal rate of substitution equals the relative price (of consumption at time t relative to consumption at time $t+1$:

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t.$$

Example: $u(C) = \ln(C)$:

$$\frac{1}{C_t} = \beta(1 + r_t)\frac{1}{C_{t+1}}$$

or

$$\frac{C_{t+1}}{C_t} = \beta(1 + r_t)$$

Example: $u(C) = \frac{1}{1-\sigma} C^{1-\sigma}$

$$C_t^{-\sigma} = \beta(1 + r_t) C_{t+1}^{-\sigma}$$

We get in this case, approximately, if we use

$$\ln(1 + r_t) = r_t$$

that

$$\ln C_{t+1} - \ln C_t = \frac{1}{\sigma} \ln \beta + \frac{1}{\sigma} r_t.$$

How consumption changes with income and interest rates

We will do this algebraically first, then graphically.

We had the first-order condition:

$$u'(C_t) - (1+r_t)\beta u'\left((1+r_t)(Y_t - C_t) + Y_{t+1}\right) = 0$$

Take the derivatives. First, holding r_t and Y_{t+1} constant, find $\dfrac{\partial C_t}{\partial Y_t}$:

$$u''(C_t)\frac{\partial C_t}{\partial Y_t} + (1+r_t)^2\beta u''(C_{t+1})\frac{\partial C_t}{\partial Y_t} - (1+r_t)^2\beta u''(C_{t+1}) = 0.$$

Solve to find: $\dfrac{\partial C_t}{\partial Y_t} = \dfrac{(1+r_t)^2\beta u''(C_{t+1})}{u''(C_t) + (1+r_t)^2\beta u''(C_{t+1})} > 0$

We see an increase in current income will increase consumption. But notice that $0 < \frac{\partial C_t}{\partial Y_t} < 1$. When current income increases, current consumption rises, but so does saving.

Suppose we learn at time t that Y_{t+1} will change. Now hold r_t and Y_t constant.

We find: $u''(C_t)\frac{\partial C_t}{\partial Y_{t+1}} + (1+r_t)^2 \beta u''(C_{t+1})\frac{\partial C_t}{\partial Y_{t+1}} - (1+r_t)u''(C_{t+1}) = 0$.

This gives us $\frac{\partial C_t}{\partial Y_{t+1}} = \frac{(1+r_t)u''(C_{t+1})}{u''(C_t) + (1+r_t)^2 \beta u''(C_{t+1})} > 0$

The household can borrow at time t if Y_{t+1} rises enough.

Finally, holding income in both periods constant, what happens if the interest rate changes?

$$u''(C_t)\frac{\partial C_t}{\partial r_t} + (1+r_t)^2 \beta u''(C_{t+1})\frac{\partial C_t}{\partial r_t}$$
$$-\beta u'(C_{t+1}) - (1+r_t)\beta(Y_t - C_t)u''(C_{t+1}) = 0$$
.

Solving this, we find

$$\frac{\partial C_t}{\partial r_t} = \frac{\beta u'(C_{t+1}) + (1+r_t)\beta(Y_t - C_t)u''(C_{t+1})}{u''(C_t) + (1+r_t)^2 \beta u''(C_{t+1})}$$

The effect on consumption is ambiguous. We can divide this derivative into parts the book calls the substitution effect and the income effect:
Substitution effect:

$$\left.\frac{\partial C_t}{\partial r_t}\right|_{Substitution} = \frac{\beta u'\left(C_{t+1}\right)}{u''\left(C_t\right)+\left(1+r_t\right)^2 \beta u''\left(C_{t+1}\right)} < 0$$

Income effect:

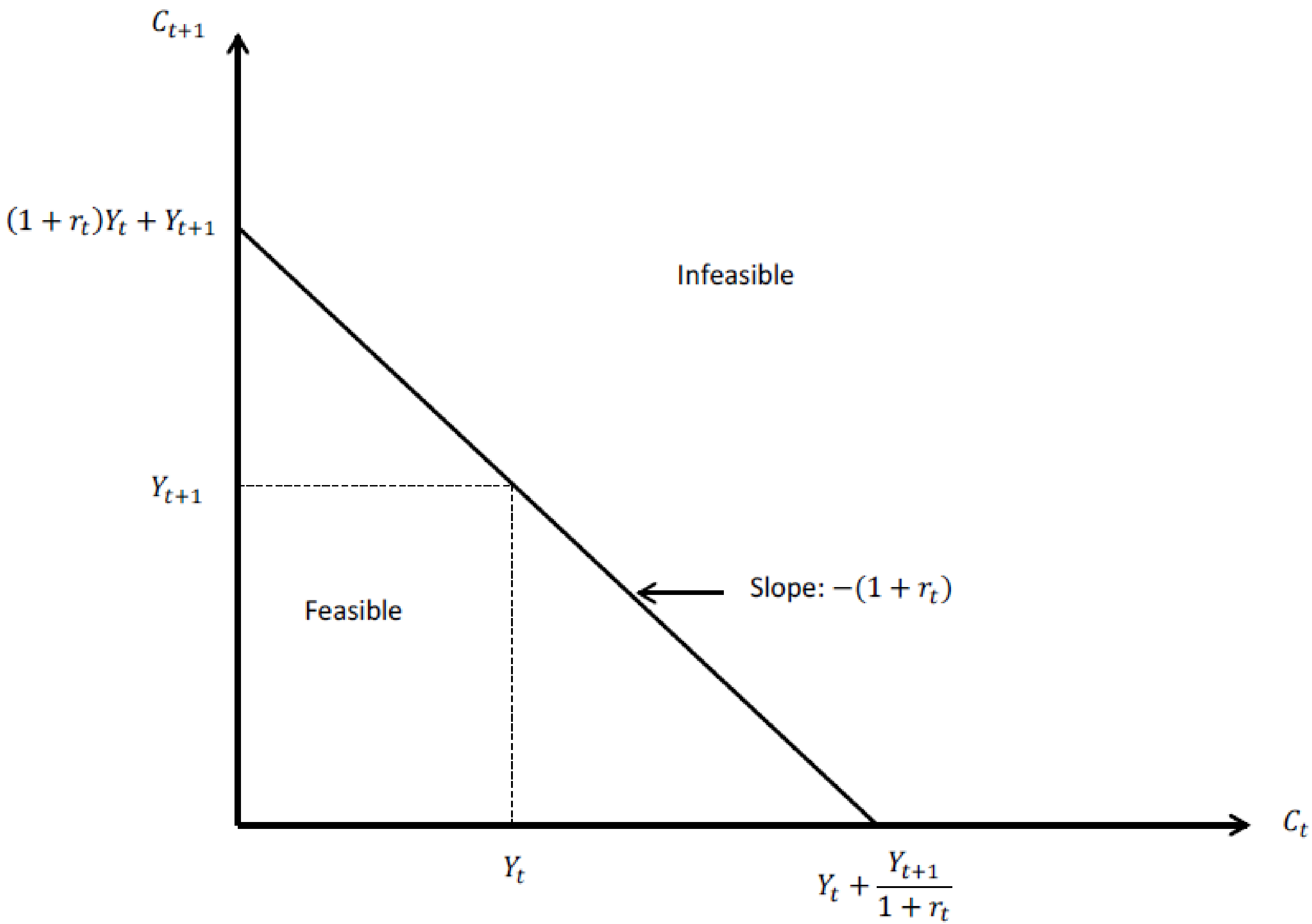
$$\left.\frac{\partial C_t}{\partial r_t}\right|_{Income} = \frac{\left(1+r_t\right)\beta\left(Y_t - C_t\right)u''\left(C_{t+1}\right)}{u''\left(C_t\right)+\left(1+r_t\right)^2 \beta u''\left(C_{t+1}\right)}$$

which is > 0 if household is a saver at time *t*, so $Y_t - C_t > 0$
but < 0 if household is a borrower at time *t*, so $Y_t - C_t < 0$

We will assume overall the substitution effect dominates.

Graphical Analysis: Indifference Curves and Budget Lines

Equation of budget line: $C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1}$



Indifference curves are combinations of current and future consumption that hold utility at a constant level:

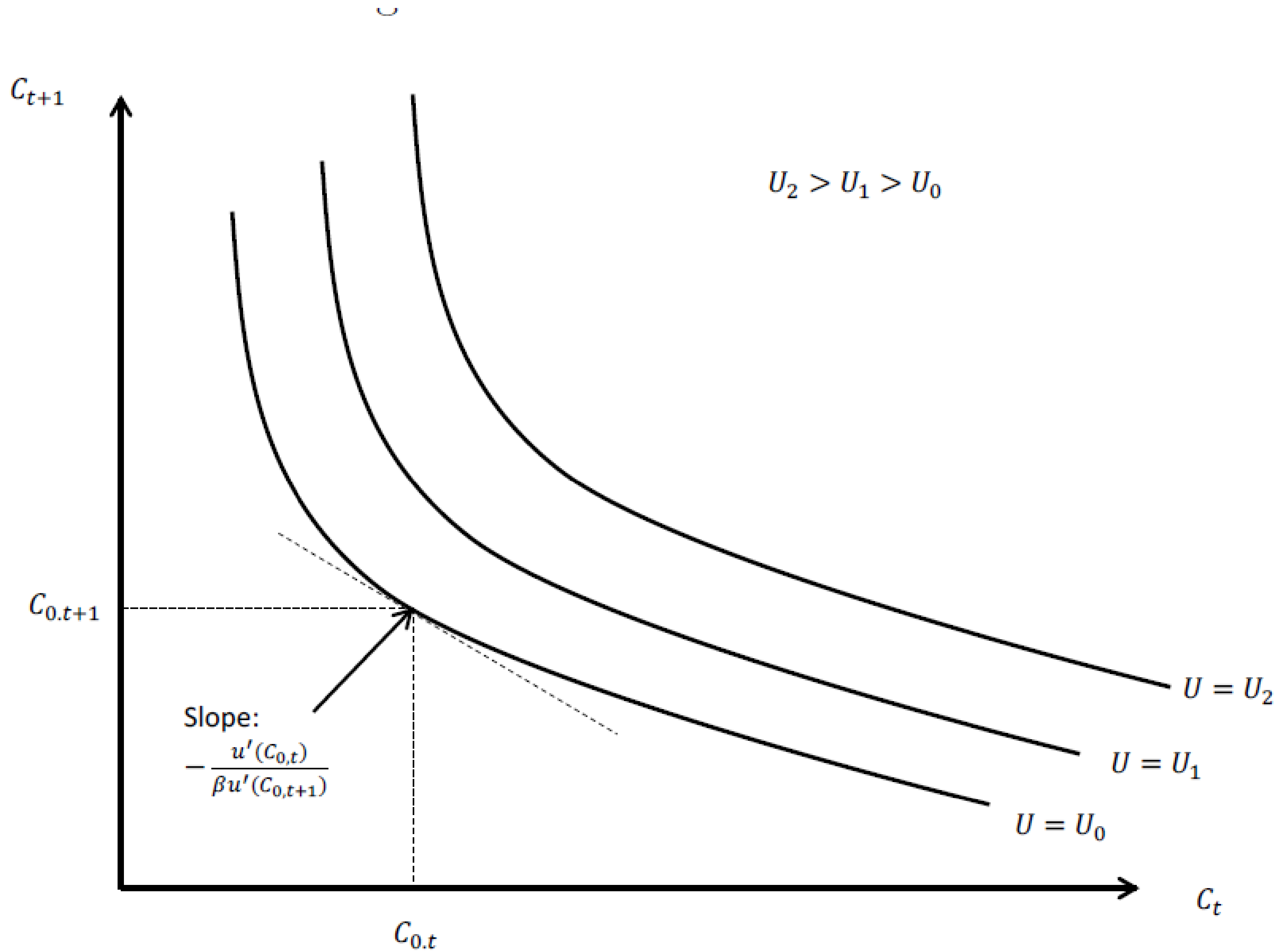
$$U_0 = u(C_{0,t}) + \beta u(C_{0,t+1})$$

Differentiate:

$$dU = u'(C_{0,t})dC_t + \beta u'(C_{0,t+1})dC_{t+1}$$

Since indifference curve holds utility constant, set $dU = 0$, and rearrange to get the equation for the slope of the indifference curve:

$$\frac{dC_{t+1}}{dC_t} = -\frac{u'(C_{0,t})}{\beta u'(C_{0,t+1})}$$



C_{t+1}
$U_2 > U_1 > U_0$
$C_{0.t+1}$
Slope:
$-\frac{u'(C_{0,t})}{\beta u'(C_{0,t+1})}$
$U = U_2$
$U = U_1$
$U = U_0$
C_t
$C_{0.t}$

Figure 9.4: An Optimal Consumption Bundle

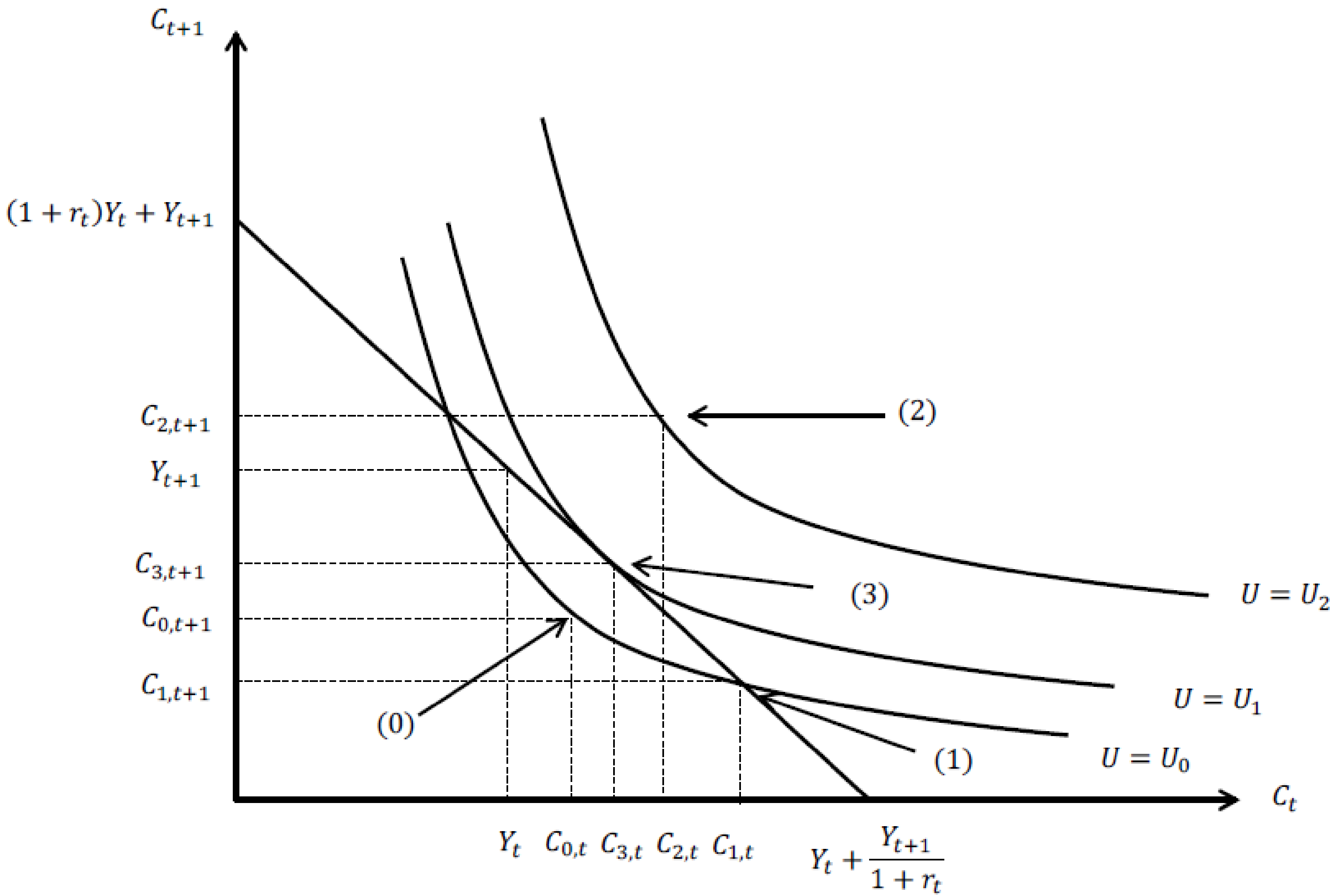


Figure 9.5: Increase in Y_t

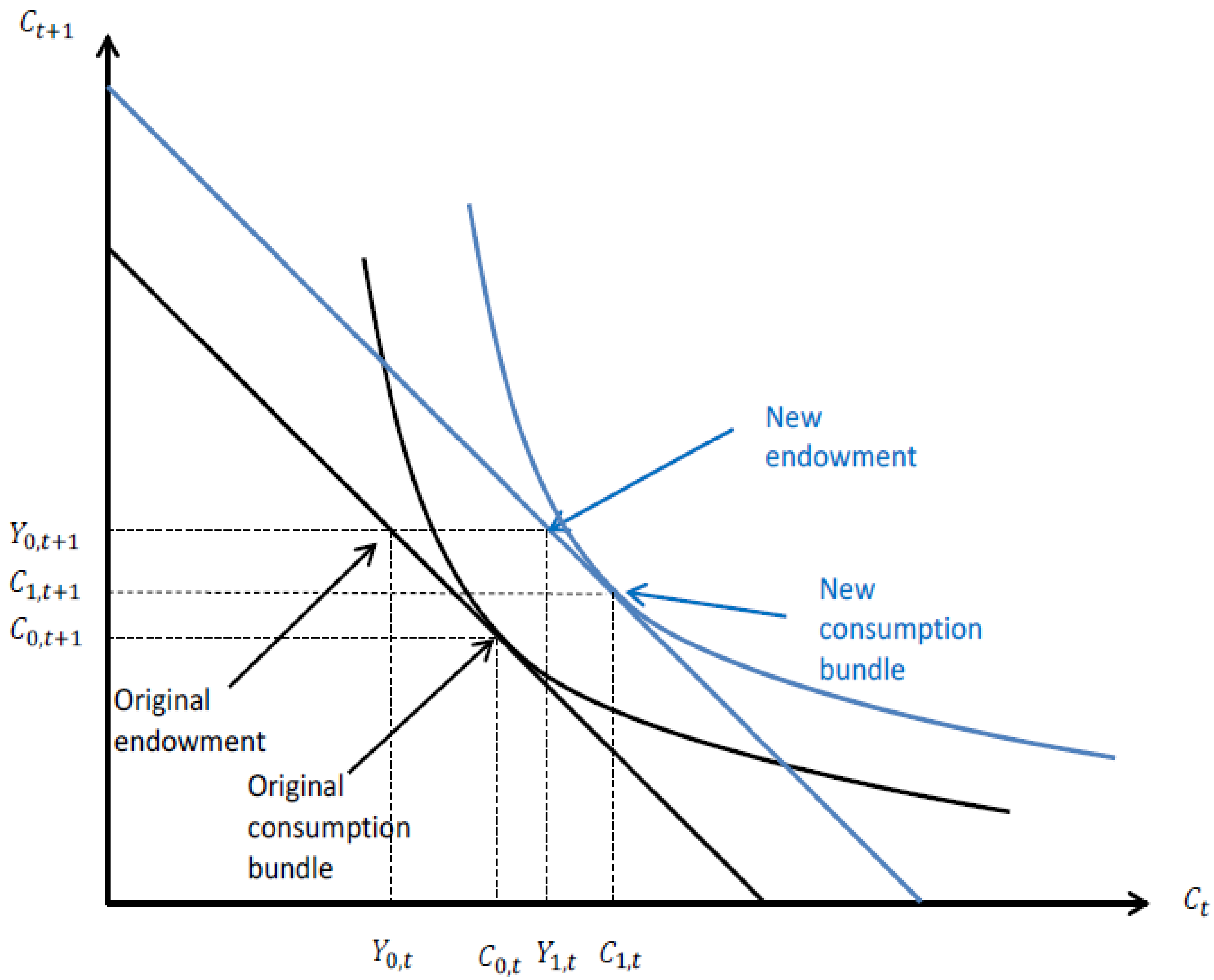


Figure 9.6: Increase in Y_{t+1}

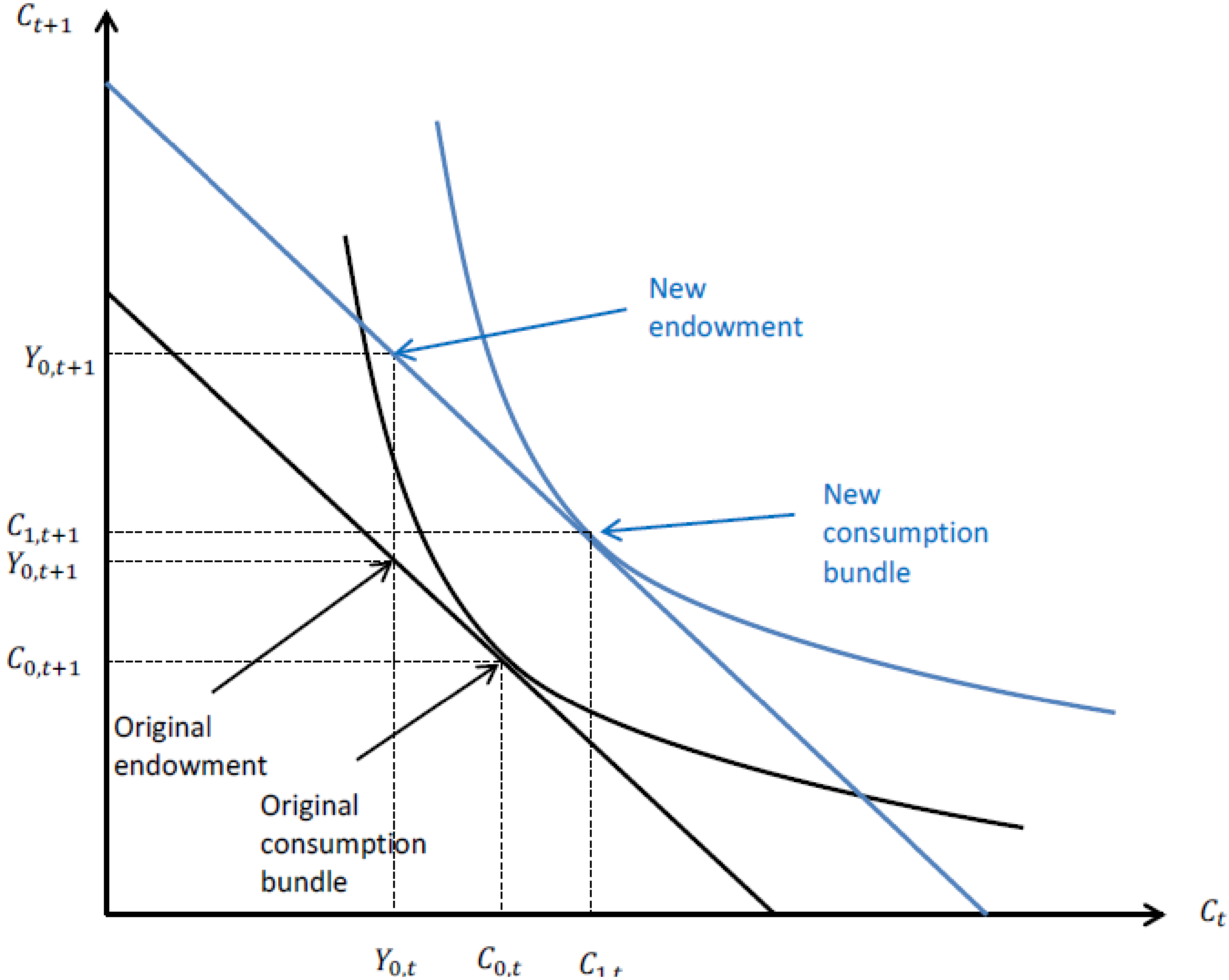


Figure 9.7: Increase in r_t and Pivot of the Budget Line

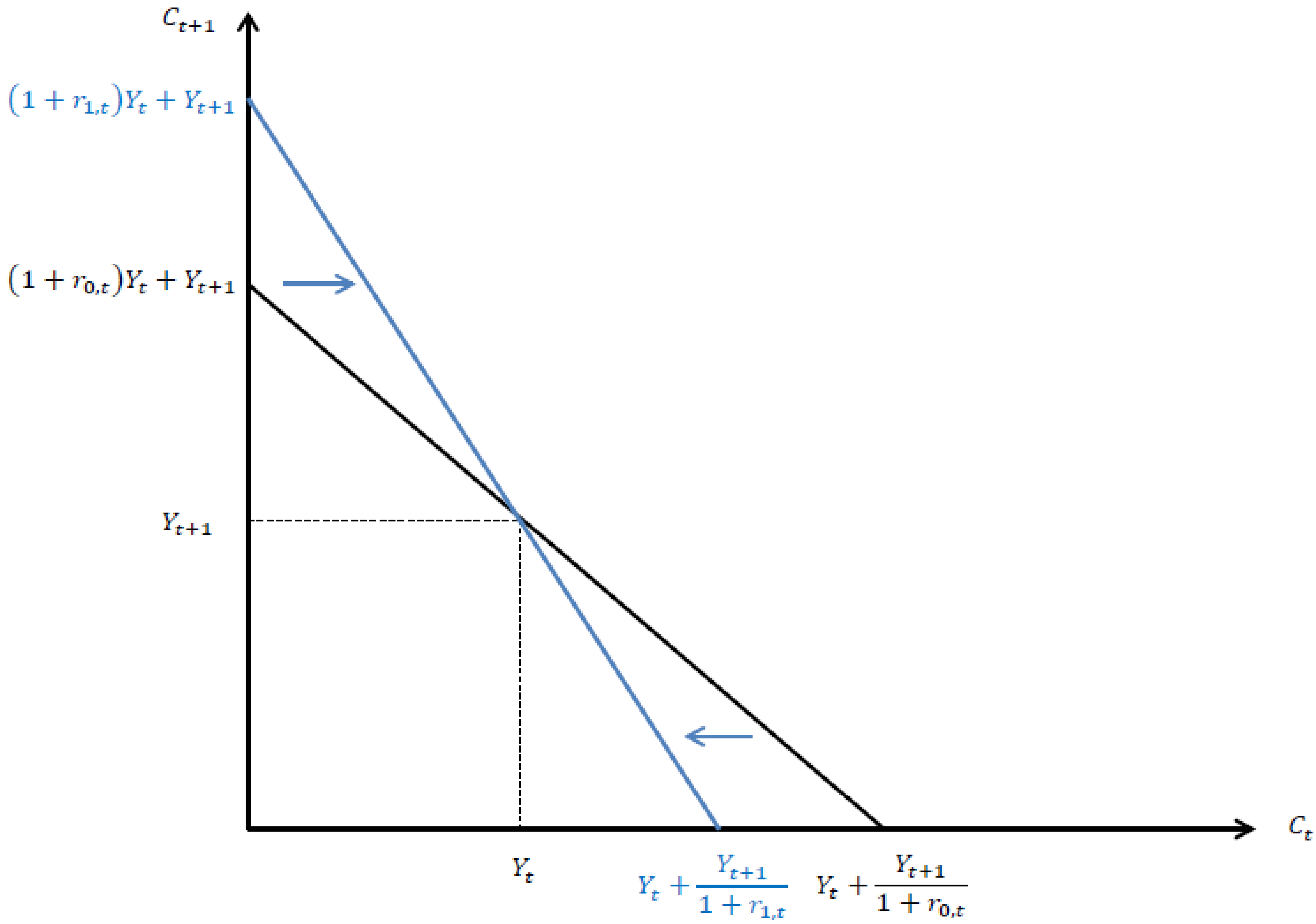


Figure 9.8: Increase in r_t: Initially a Borrower

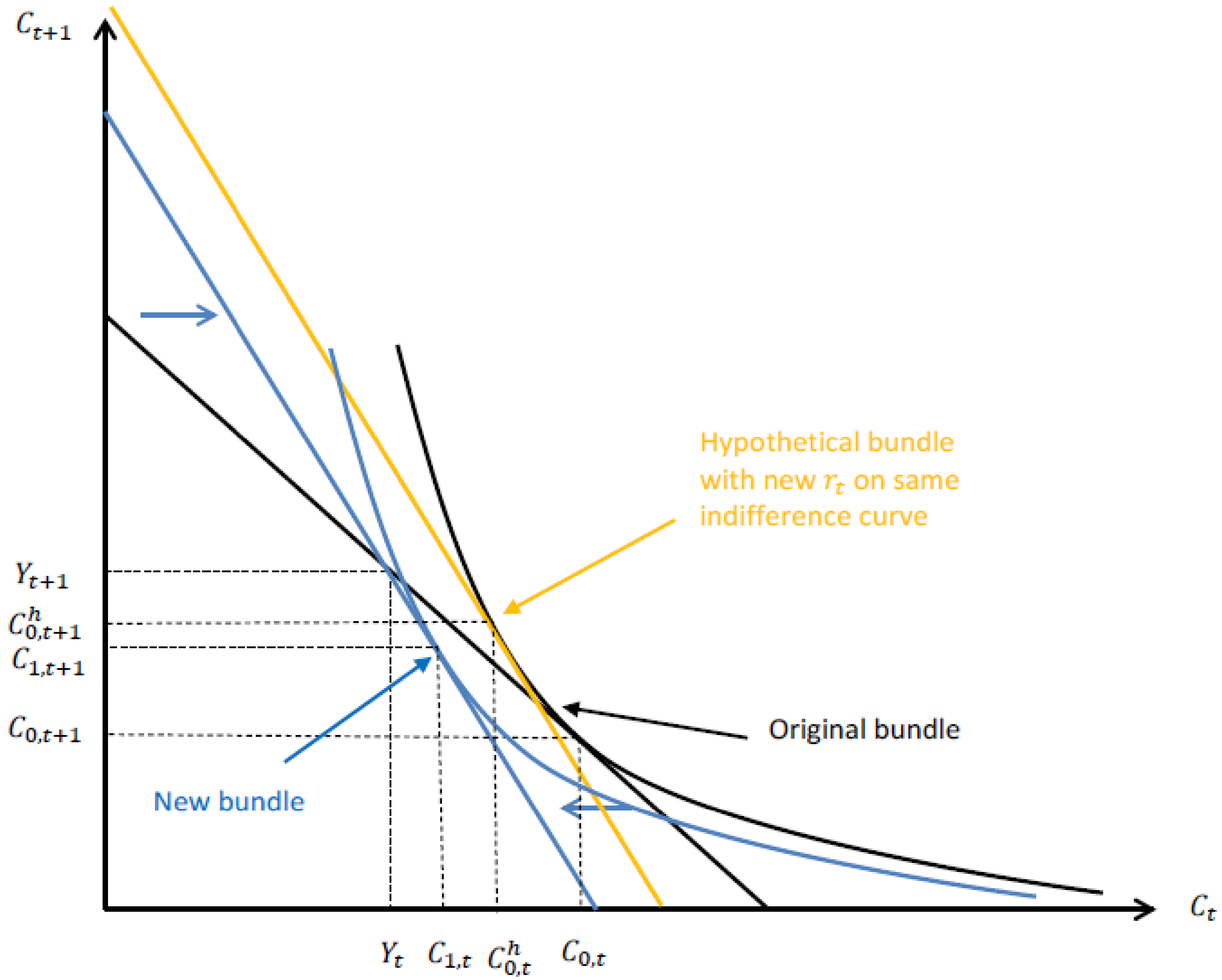


Figure 9.9: Increase in r_t: Initially a Saver

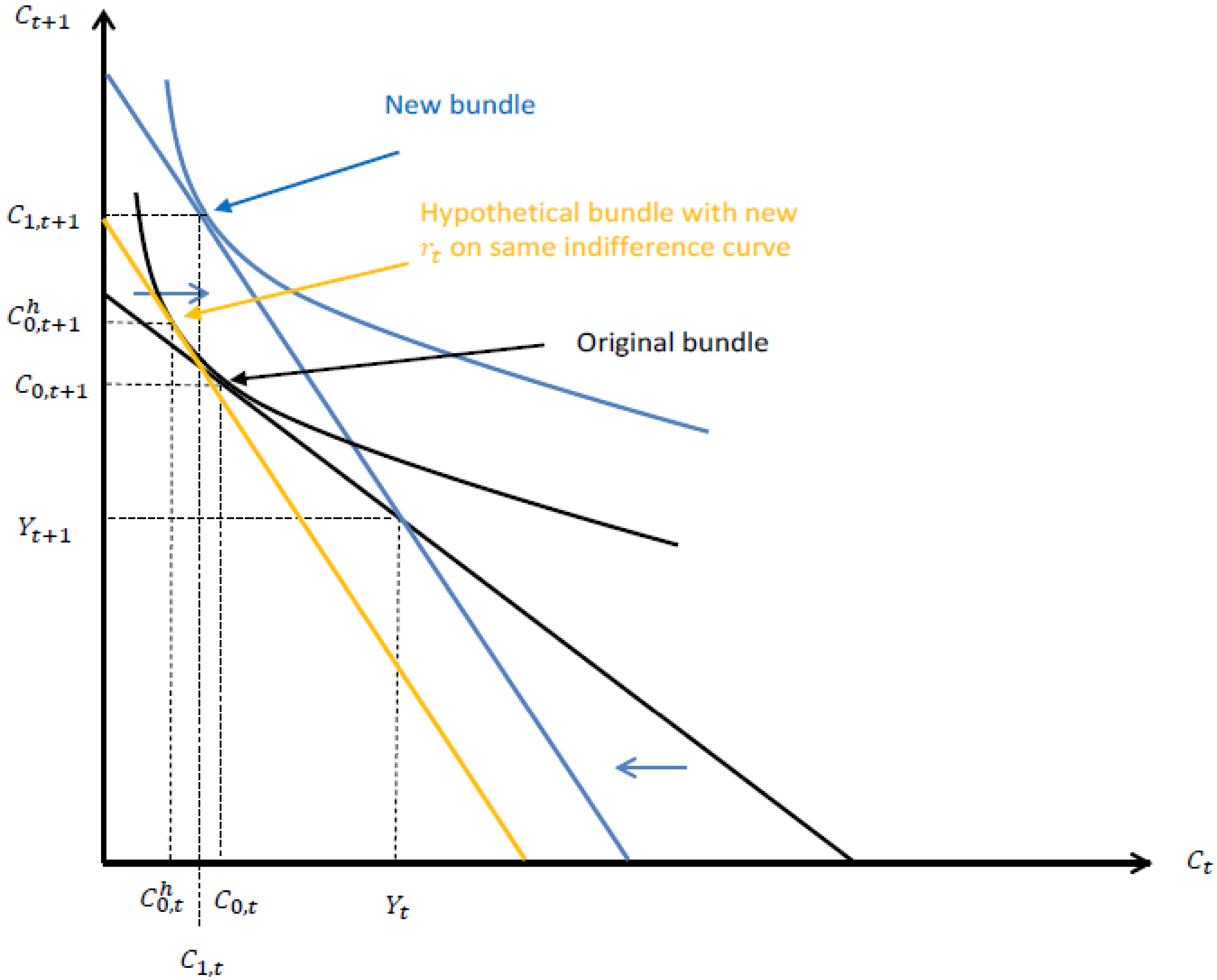


Table 9.1: Income and Substitution Effects of Higher r_t

	Substitution Effect	Income Effect	Total Effect
C_t			
Borrower	-	-	-
Saver	-	+	?
C_{t+1}			
Borrower	+	-	?
Saver	+	+	+

Assume the substitution effect dominates so

$$C_t = C^d(\underset{+}{Y_t}, \underset{+}{Y_{t+1}}, \underset{-}{r_t})$$

Example: $u(C) = \ln(C)$

$$C_t = \frac{1}{1+\beta}\left[Y_t + \frac{Y_{t+1}}{1+r_t}\right]$$

$$\frac{\partial C_t}{\partial Y_t} = \frac{1}{1+\beta}$$

$$\frac{\partial C_t}{\partial Y_{t+1}} = \frac{1}{1+\beta}\frac{1}{1+r_t}$$

$$\frac{\partial C_t}{\partial r_t} = -\frac{Y_{t+1}}{1+\beta}(1+r_t)^{-2}$$

Permanent Income Changes

Suppose that when Y_t rises, we know also that Y_{t+1} will increase the same amount. The income increase is *permanent*.

$$\frac{dC_t}{dY_t} = \frac{\partial C_t}{\partial Y_t} + \frac{\partial C_{t+1}}{\partial Y_{t+1}} > \frac{\partial C_t}{\partial Y_t}.$$

The effect of a permanent change in income is greater than the effect of a *transitory* change.

Similarly, a permanent cut in taxes has a larger effect on consumption than a transitory change, according to the model.

Taxes

Assume “lump-sum” taxes, which work just like a decrease in the household’s income:

$$C_t + S_t \leq Y_t - T_t$$

$$C_{t+1} + S_{t+1} \leq Y_{t+1} - T_{t+1} + (1 + r_t)S_t$$

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t - T_t + \frac{Y_{t+1} - T_{t+1}}{1 + r_t}$$

Does the empirical evidence support the claim that a transitory tax cut has a smaller effect on consumption than a permanent tax cut?