The model of an optimizing household forms the basis for macroeconomic models of consumption and saving.

Dynamic

Two-period: Why?

Representative household

Periods $t$ and $t + 1$.

Household chooses $C_t$, $S_t$, $C_{t+1}$, $S_{t+1}$ given $Y_t$, $Y_{t+1}$, $r_t$

Exogenous vs. endogenous
Household’s budget constraints:

\[ C_t + S_t \leq Y_t \]
\[ C_{t+1} + S_{t+1} \leq Y_{t+1} + (1 + r_t)S_t \]

The latter can be written as:

\[ C_{t+1} + S_{t+1} - S_t \leq Y_{t+1} + r_t S_t \]

(“saving” and “savings”)

Under optimality, we will have \( S_{t+1} = 0 \) and budget constraint will hold with equality.
We have then:

\[ C_t + S_t = Y_t \]
\[ C_{t+1} = Y_{t+1} + (1 + r_t) S_t \]

Solve out for \( S_t \) and rearrange to get a single constraint:

\[ C_t + \frac{C_{t+1}}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t} \]

“present value” and “current value”
Households maximize

\[ U = u(C_t) + \beta u(C_{t+1}), \quad 0 \leq \beta < 1 \]

Marginal utility is positive, always:

\[ u'(\cdot) > 0 \]

Diminishing marginal utility:

\[ u''(C_t) < 0 \]
Some example utility functions:

\[ u(C_t) = \theta C_t, \quad \theta > 0 \]
\[ u(C_t) = C_t - \frac{\theta}{2} C_t^2, \quad \theta > 0 \]
\[ u(C_t) = \ln C_t \]
\[ u(C_t) = \frac{C_t^{1-\sigma} - 1}{1 - \sigma} = \frac{C_t^{1-\sigma}}{1 - \sigma} - \frac{1}{1 - \sigma}, \quad \sigma > 0 \]

The first one is not concave (the second derivative is zero.)

The quadratic utility has the problem that the first derivative turns negative after a certain point.
Figure 9.1: Utility and Marginal Utility

\[ u(C_t) \]

\[ u'(C_t) \]
The household’s problem:

$$\max_{C_t, C_{t+1}} U = u(C_t) + \beta u(C_{t+1})$$

subject to:

$$C_t + \frac{C_{t+1}}{1 + r_t} = Y_t + \frac{Y_{t+1}}{1 + r_t}.$$ 

We could use a Lagrangian to set up this problem. Instead, we will substitute out for $C_{t+1}$. That is, we will solve for $C_{t+1}$ from the constraint:

$$C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1}.$$
Then substitute this solution into the utility function to get an unconstrained maximization problem:

\[
\max_{C_t} \quad U = u(C_t) + \beta u ((1 + r_t)(Y_t - C_t) + Y_{t+1})
\]

The first-order condition is:

\[
\frac{\partial U}{\partial C_t} = u'(C_t) - (1 + r_t) \beta u'((1 + r_t)(Y_t - C_t) + Y_{t+1}) = 0
\]

But since \( C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1} \), we can write

\[
u'(C_t) - (1 + r_t) \beta u'(C_{t+1}) = 0
\]
It is intuitive to write this as:

\[ u'(C_t) = \beta (1 + r_t) u'(C_{t+1}) \]

This is called the Euler equation.

We could write this as the marginal rate of substitution equals the relative price (of consumption at time \( t \) relative to consumption at time \( t + 1 \)):

\[ \frac{u'(C_t)}{\beta u'(C_{t+1})} = 1 + r_t. \]
Example: $u(C) = \ln(C)$:

$$\frac{1}{C_t} = \beta (1 + r_t) \frac{1}{C_{t+1}}$$

or

$$\frac{C_{t+1}}{C_t} = \beta (1 + r_t)$$
Example: \( u(C) = \frac{1}{1-\sigma} C^{1-\sigma} \)

\( C_t^{\sigma} = \beta (1 + r_t) C_{t+1}^{\sigma} \)

We get in this case, approximately, if we use

\( \ln(1 + r_t) = r_t \)

that

\( \ln C_{t+1} - \ln C_t = \frac{1}{\sigma} \ln \beta + \frac{1}{\sigma} r_t. \)
How consumption changes with income and interest rates

We will do this algebraically first, then graphically.

We had the first-order condition:

\[ u'(C_t) - (1 + r_t) \beta u' \left( (1 + r_t)(Y_t - C_t) + Y_{t+1} \right) = 0 \]

Take the derivatives. First, holding \( r_t \) and \( Y_{t+1} \) constant, find \( \frac{\partial C_t}{\partial Y_t} \):

\[ u''(C_t) \frac{\partial C_t}{\partial Y_t} + (1 + r_t)^2 \beta u''(C_{t+1}) \frac{\partial C_t}{\partial Y_t} - (1 + r_t)^2 \beta u''(C_{t+1}) = 0. \]

Solve to find:

\[ \frac{\partial C_t}{\partial Y_t} = \frac{(1 + r_t)^2 \beta u''(C_{t+1})}{u''(C_t) + (1 + r_t)^2 \beta u''(C_{t+1})} > 0 \]
We see an increase in current income will increase consumption. But notice that \( 0 < \frac{\partial C_t}{\partial Y_t} < 1 \). When current income increases, current consumption rises, but so does saving.

Suppose we learn at time \( t \) that \( Y_{t+1} \) will change. Now hold \( r_t \) and \( Y_t \) constant.

We find: 
\[
u''(C_t) \frac{\partial C_t}{\partial Y_{t+1}} + (1 + r_t)^2 \beta u''(C_{t+1}) \frac{\partial C_t}{\partial Y_{t+1}} - (1 + r_t)u''(C_{t+1}) = 0.
\]

This gives us 
\[
\frac{\partial C_t}{\partial Y_{t+1}} = \frac{(1 + r_t)u''(C_{t+1})}{u''(C_t) + (1 + r_t)^2 \beta u''(C_{t+1})} > 0
\]

The household can borrow at time \( t \) if \( Y_{t+1} \) rises enough.
Finally, holding income in both periods constant, what happens if the interest rate changes?

\[ u''(C_t) \frac{\partial C_t}{\partial r_t} + (1 + r_t)^2 \beta u''(C_{t+1}) \frac{\partial C_t}{\partial r_t} + \beta u'(C_{t+1}) - (1 + r_t) \beta (Y_t - C_t) u''(C_{t+1}) = 0 \]

Solving this, we find

\[ \frac{\partial C_t}{\partial r_t} = \frac{\beta u'(C_{t+1}) + (1 + r_t) \beta (Y_t - C_t) u''(C_{t+1})}{u''(C_t) + (1 + r_t)^2 \beta u''(C_{t+1})} \]

The effect on consumption is ambiguous. We can divide this derivative into parts the book calls the substitution effect and the income effect:

Substitution effect:
\[
\left. \frac{\partial C_t}{\partial r_t} \right|_{Substitution} = \frac{\beta u'(C_{t+1})}{u''(C_t) + (1 + r_t)^2 \beta u''(C_{t+1})} < 0
\]

Income effect:

\[
\left. \frac{\partial C_t}{\partial r_t} \right|_{Income} = \frac{(1 + r_t) \beta (Y_t - C_t) u''(C_{t+1})}{u''(C_t) + (1 + r_t)^2 \beta u''(C_{t+1})}
\]

which is > 0 if household is a saver at time \( t \), so \( Y_t - C_t > 0 \)
but < 0 if household is a borrower at time \( t \), so \( Y_t - C_t < 0 \)

We will assume overall the substitution effect dominates.
Graphical Analysis: Indifference Curves and Budget Lines

Equation of budget line: $C_{t+1} = (1 + r_t)(Y_t - C_t) + Y_{t+1}$
Indifference curves are combinations of current and future consumption that hold utility at a constant level:

\[ U_0 = u(C_{0,t}) + \beta u(C_{0,t+1}) \]

Differentiate:

\[ dU = u'(C_{0,t}) dC_t + \beta u'(C_{0,t+1}) dC_{t+1} \]

Since indifference curve holds utility constant, set \( dU = 0 \), and rearrange to get the equation for the slope of the indifference curve:

\[ \frac{dC_{t+1}}{dC_t} = -\frac{u'(C_{0,t})}{\beta u'(C_{0,t+1})} \]
Figure 9.4: An Optimal Consumption Bundle

\[ (1 + r_t)Y_t + Y_{t+1} \]

\[ C_{t+1} \]

\[ Y_{t+1} \]

\[ C_{2,t+1} \]

\[ C_{3,t+1} \]

\[ C_{0,t+1} \]

\[ C_{1,t+1} \]

\[ Y_t \]

\[ C_{0,t} \]

\[ C_{3,t} \]

\[ C_{2,t} \]

\[ C_{1,t} \]

\[ Y_t + \frac{Y_{t+1}}{1 + r_t} \]

\[ U = U_2 \]

\[ U = U_1 \]

\[ U = U_0 \]

(0)

(1)

(2)

(3)
Figure 9.5: Increase in $Y_t$

The graph illustrates the increase in consumption ($C_{t+1}$) with respect to time ($Y_t$). The original endowment is marked by $Y_{0,t+1}$ and $C_{0,t+1}$, and the new endowment is shown by $Y_{1,t+1}$ and $C_{1,t+1}$. The diagram also highlights the original consumption bundle and the new consumption bundle.
Figure 9.6: Increase in $Y_{t+1}$
Figure 9.7: Increase in $r_t$ and Pivot of the Budget Line

\[ (1 + r_{1,t})Y_t + Y_{t+1} \]

\[ (1 + r_{0,t})Y_t + Y_{t+1} \]

\[ Y_t \]

\[ Y_t + \frac{Y_{t+1}}{1 + r_{1,t}} \]

\[ Y_t + \frac{Y_{t+1}}{1 + r_{0,t}} \]
Figure 9.8: Increase in $r_t$: Initially a Borrower

- Hypothetical bundle with new $r_t$ on same indifference curve
- Original bundle
- New bundle
Figure 9.9: Increase in $r_t$: Initially a Saver
Table 9.1: Income and Substitution Effects of Higher $r_t$

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<thead>
<tr>
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<th>Substitution Effect</th>
<th>Income Effect</th>
<th>Total Effect</th>
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<tr>
<td>$C_t$</td>
<td></td>
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<tr>
<td>Borrower</td>
<td>-</td>
<td>-</td>
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<tr>
<td>Saver</td>
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</tr>
<tr>
<td>$C_{t+1}$</td>
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<tr>
<td>Borrower</td>
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<td>-</td>
<td>?</td>
</tr>
<tr>
<td>Saver</td>
<td>+</td>
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</tbody>
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Assume the substitution effect dominates so

$$C_t = C^d(Y_t, Y_{t+1}, r_t)$$
Example: $u(C) = \ln(C)$

$$C_t = \frac{1}{1 + \beta} \left[ Y_t + \frac{Y_{t+1}}{1 + r_t} \right]$$

$$\frac{\partial C_t}{\partial Y_t} = \frac{1}{1 + \beta}$$

$$\frac{\partial C_t}{\partial Y_{t+1}} = \frac{1}{1 + \beta} \frac{1}{1 + r_t}$$

$$\frac{\partial C_t}{\partial r_t} = -\frac{Y_{t+1}}{1 + \beta} (1 + r_t)^{-2}$$
Permanent Income Changes

Suppose that when $Y_t$ rises, we know also that $Y_{t+1}$ will increase the same amount. The income increase is permanent.

\[ \frac{dC_t}{dY_t} = \frac{\partial C_t}{\partial Y_t} + \frac{\partial C_{t+1}}{\partial Y_{t+1}} > \frac{\partial C_t}{\partial Y_t}. \]

The effect of a permanent change in income is greater than the effect of a transitory change.

Similarly, a permanent cut in taxes has a larger effect on consumption than a transitory change, according to the model.
Taxes

Assume “lump-sum” taxes, which work just like a decrease in the household’s income:

\[
C_t + S_t \leq Y_t - T_t
\]

\[
C_{t+1} + S_{t+1} \leq Y_{t+1} - T_{t+1} + (1 + r_t) S_t
\]

\[
C_t + \frac{C_{t+1}}{1 + r_t} = Y_t - T_t + \frac{Y_{t+1} - T_{t+1}}{1 + r_t}
\]

Does the empirical evidence support the claim that a transitory tax cut has a smaller effect on consumption than a permanent tax cut?