Econ 702

Macroeconomics I

Charles Engel and Menzie Chinn

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Lecture 7: Optimal Consumption and Saving, part 2

In real life we are not certain about future income.

Solve simple example where future income is uncertain.

Suppose $U = \ln\left(C_t\right) + \ln\left(C_{t+1}\right)$

Notice we have set $\beta = 1$. This is just to make the problem simpler. Suppose we also set $r_t = 0$.

With no uncertainty, the problem is to choose C_t and C_{t+1} to maximize U subject to:

$$C_t + C_{t+1} = Y_t + Y_{t+1}$$

The problem is easy to solve. From the budget constraint,

$$C_{t+1} = Y_t + Y_{t+1} - C_t$$

Then we can choose C_t to maximize

$$U = \ln(C_t) + \ln(Y_t + Y_{t+1} - C_t)$$

From the first-order conditions we get:

$$\frac{1}{C_t} - \frac{1}{Y_t + Y_{t+1} - C_t} = 0,$$

We solve for this to find $C_t = \frac{1}{2}(Y_t + Y_{t+1})$ and $C_{t+1} = \frac{1}{2}(Y_t + Y_{t+1})$

Now suppose there is uncertainty about next period's income. We could have $Y_{t+1} = Y_{t+1}^h$ or $Y_{t+1} = Y_{t+1}^\ell$, with $Y_{t+1}^\ell < Y_{t+1}^h$.

Let's say these outcomes each have probability ½.

Next period's consumption, C_{t+1}, is therefore uncertain. C_{t+1} will be different when $Y_{t+1} = Y_{t+1}^h$ than when $Y_{t+1} = Y_{t+1}^\ell$.

The household's objective is to maxmize:

$$U = u\left(C_t\right) + E_t u\left(C_{t+1}\right)$$

What is the meaning of $E_t u\left(C_{t+1}\right)$?

Random variables and expectations

We think of consumption and output here as random variables. A simple example of a variable we might treat as random is the number assigned to the role of the dice.

Technically, a random variable is a function that assigns a numerical value to a "state of nature." For example, a coin flip might turn up heads or tails. We can assign the outcome the number one if it is heads and zero if it is tails.

The probability of outcomes for random variables is determined by the probability distribution and probability density. If we roll a “fair” dice, it the six possible outcomes are equally likely:

Outcome	Probability
1	1/6
2	1/6
3	1/6
4	1/6
5	1/6
6	1/6

The mean or expected value of the dice role is given by the sum of the products of the outcome and the probability:

$$1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=3.5.$$

Suppose an economic model included some household that determined its consumption by looking at the role of the dice. How would we model the household's expectation of the dice roll?

We will assume "rational expectations". In this case, people expect the outcome of the dice roll to be 3.5. We will build models with random variables in them. In our models, we will assume that people's expectations of the outcomes of those random variables are equal to the mean or expected value (in the statistical sense) of the random variable.

This is not an assumption about what outcome the agent expects or what outcome the agent thinks is most likely. Think of the dice roll. Each of the outcomes, 1 through 6, is equally likely. When we say the expected value is 3.5, we don't mean that the agent actually thinks the dice is going to be rolled and somehow a 3.5 is going to come up.

In our case, let's call C_{t+1}^h the level of consumption the household ends up choosing when $Y_{t+1} = Y_{t+1}^h$, and C_{t+1}^ℓ the level chosen when $Y_{t+1} = Y_{t+1}^\ell$.

In our example, when $Y_{t+1} = Y_{t+1}^h$ and $Y_{t+1} = Y_{t+1}^\ell$ each have probability ½, and utiilty is logarithmic,

$$E_t u\left(C_{t+1}\right) = \frac{1}{2}\ln\left(C_{t+1}^h\right) + \frac{1}{2}\ln\left(C_{t+1}^\ell\right)$$

The budget constraints faced by the household are:

$$C_t + S_t = Y_t$$
$$C_{t+1}^h = Y_{t+1}^h + S_t$$
$$C_{t+1}^\ell = Y_{t+1}^\ell + S_t$$

Budget constraint must hold regardless of state of income.

We can rewrite the problem now so that the household's objective is to choose C_t to maximize:

$$U = \ln\left(C_t\right) + \frac{1}{2}\ln\left(Y_{t+1}^h + Y_t - C_t\right) + \frac{1}{2}\ln\left(Y_{t+1}^\ell + Y_t - C_t\right)$$

The first-order condition is:

$$\frac{1}{C_t} - \frac{1}{2}\frac{1}{\left(Y_{t+1}^h + Y_t - C_t\right)} - \frac{1}{2}\frac{1}{\left(Y_{t+1}^\ell + Y_t - C_t\right)} = 0$$

Working this out, we get a quadratic equation for C_t:

$$2\left(Y_{t+1}^h + Y_t\right)\left(Y_{t+1}^\ell + Y_t\right) - 3C\left(Y_{t+1}^h + Y_{t+1}^\ell + 2Y_t\right) + 4C_t^2 = 0$$

The solution to this equation, using the quadratic formula, is:

$$C_t = \frac{3\left(Y_{t+1}^h + Y_{t+1}^\ell + 2Y\right) - \sqrt{9\left(Y_{t+1}^h + Y_{t+1}^\ell + 2Y\right)^2 - 32\left(Y_{t+1}^h + Y\right)\left(Y_{t+1}^\ell + Y\right)}}{8}$$

That seems like a mess. But here is the point.

Let's call the solution to the optimal consumption problem $\tilde{C}_t$, so

$$\tilde{C}_t = \frac{3\left(Y_{t+1}^h + Y_{t+1}^\ell + 2Y\right) - \sqrt{9\left(Y_{t+1}^h + Y_{t+1}^\ell + 2Y\right)^2 - 32\left(Y_{t+1}^h + Y\right)\left(Y_{t+1}^\ell + Y\right)}}{8}$$

When there was no uncertainty, the optimal consumption was:

$$C_t = \frac{Y_t + Y_{t+1}}{2}.$$

A reasonable question to ask is how does uncertainty affect our saving? If our only concern was our expected income, we might guess that the solution to the optimal consumption problem was to consume

$$\frac{Y_t + E_t Y_{t+1}}{2}, \text{ where } E_t Y_{t+1} = \frac{1}{2} Y_{t+1}^h + \frac{1}{2} Y_{t+1}^\ell$$

But in fact, $\tilde{C}_t < \frac{Y_t + E_t Y_{t+1}}{2}$ (as you will show on your next homework!)

The book shows the following result graphically:

If $u'''(c_t) > 0$, and there is an increase in uncertainty about future income that does not change $E_t Y_{t+1}$, then consumption must fall.

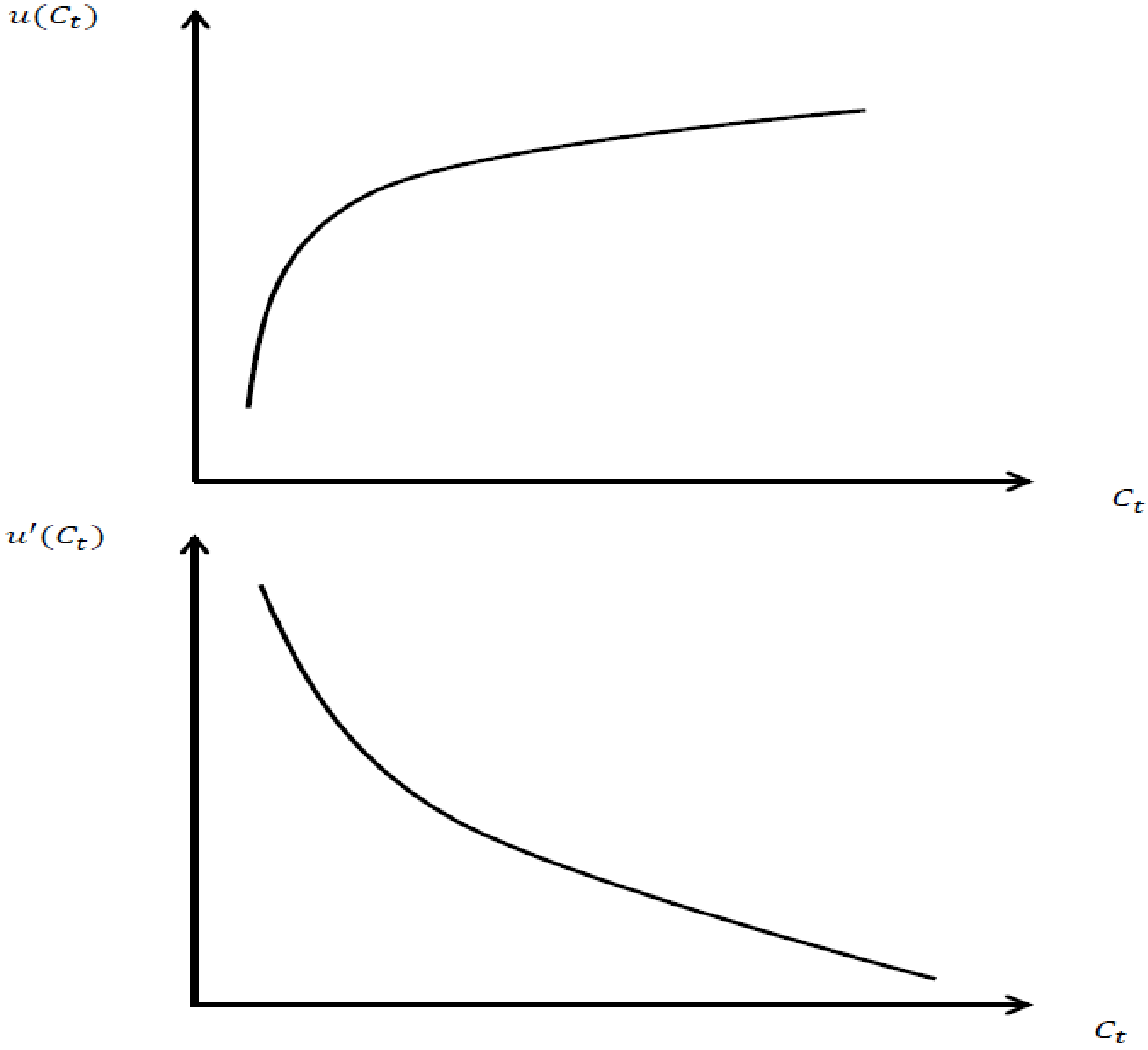
This is called “precautionary saving”.

There are two questions here:

What does it mean for $u'''(c_t) > 0$?

What is an increase in uncertainty?

Figure 9.1: Utility and Marginal Utility



Variance

The variance is an expected value.

The variance of the random variable is $E\left(X-E(X)\right)^2$. From every possible realization of X, subtract off the mean, $E(X)$. Square the difference, to get $\left(X-E(X)\right)^2$.

Then get the expectation of $\left(X-E(X)\right)^2$ by multiplying each value of $\left(X-E(X)\right)^2$ by its probability and summing up, to get $E\left(X-E(X)\right)^2$.

In the dice roll example, here is how we calculate the variance:

Outcome	Probability	$(X - E((X))^2$
1	1/6	$(2.5)^2 = 6.25$
2	1/6	$(1.5)^2 = 2.25$
3	1/6	$(0.5)^2 = 0.25$
4	1/6	$(0.5)^2 = 0.25$
5	1/6	$(1.5)^2 = 2.25$
6	1/6	$(2.5)^2 = 6.25$

The variance is given by

$$\frac{1}{6}(6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25) = \frac{17.5}{6}.$$

The standard deviation is the square root of the variance.

Figure 9.12: Expected Marginal Utility and Marginal Utility of Expected Consumption

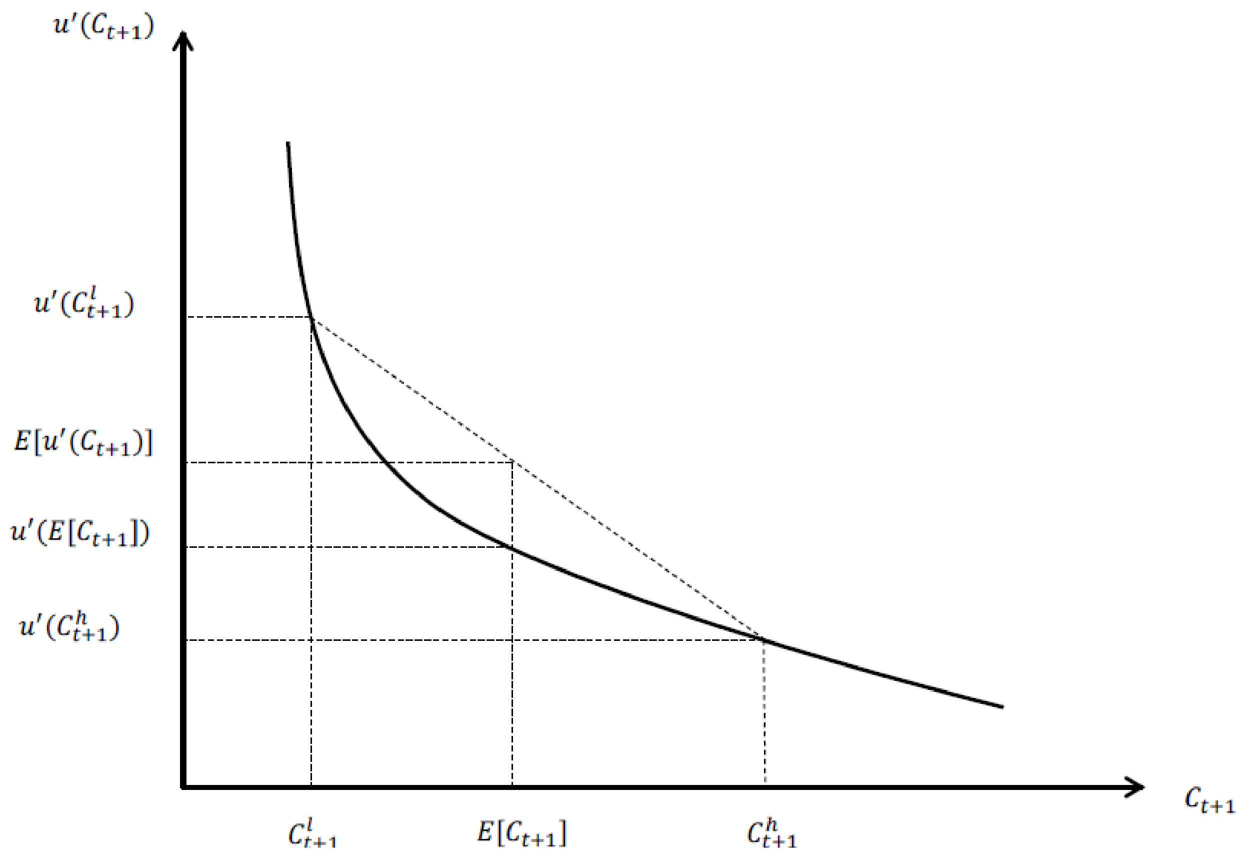
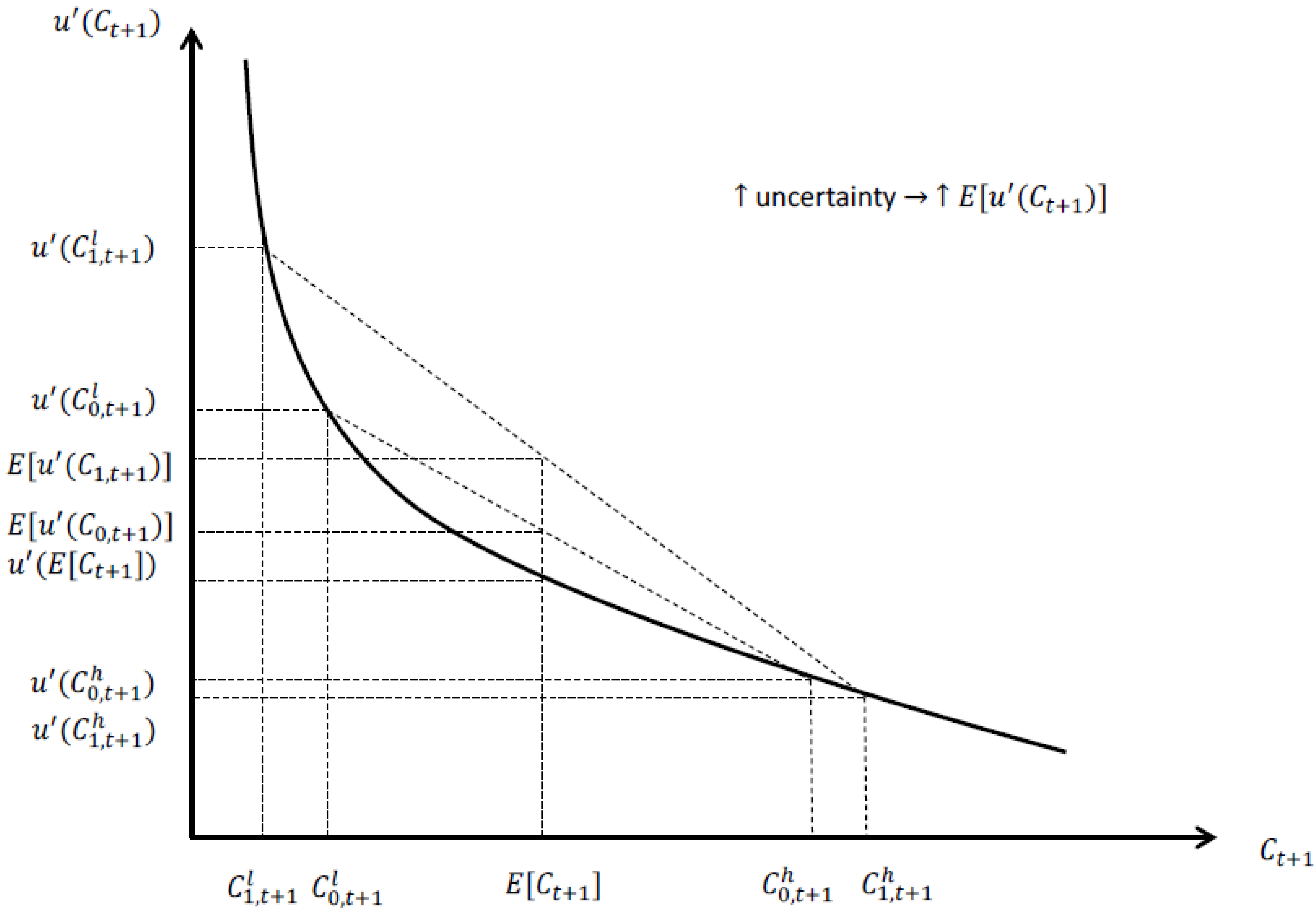


Figure 9.13: An Increase in Uncertainty



Borrowing Constraints

Figure 9.14: Borrowing Constraint: $S_t \geq 0$

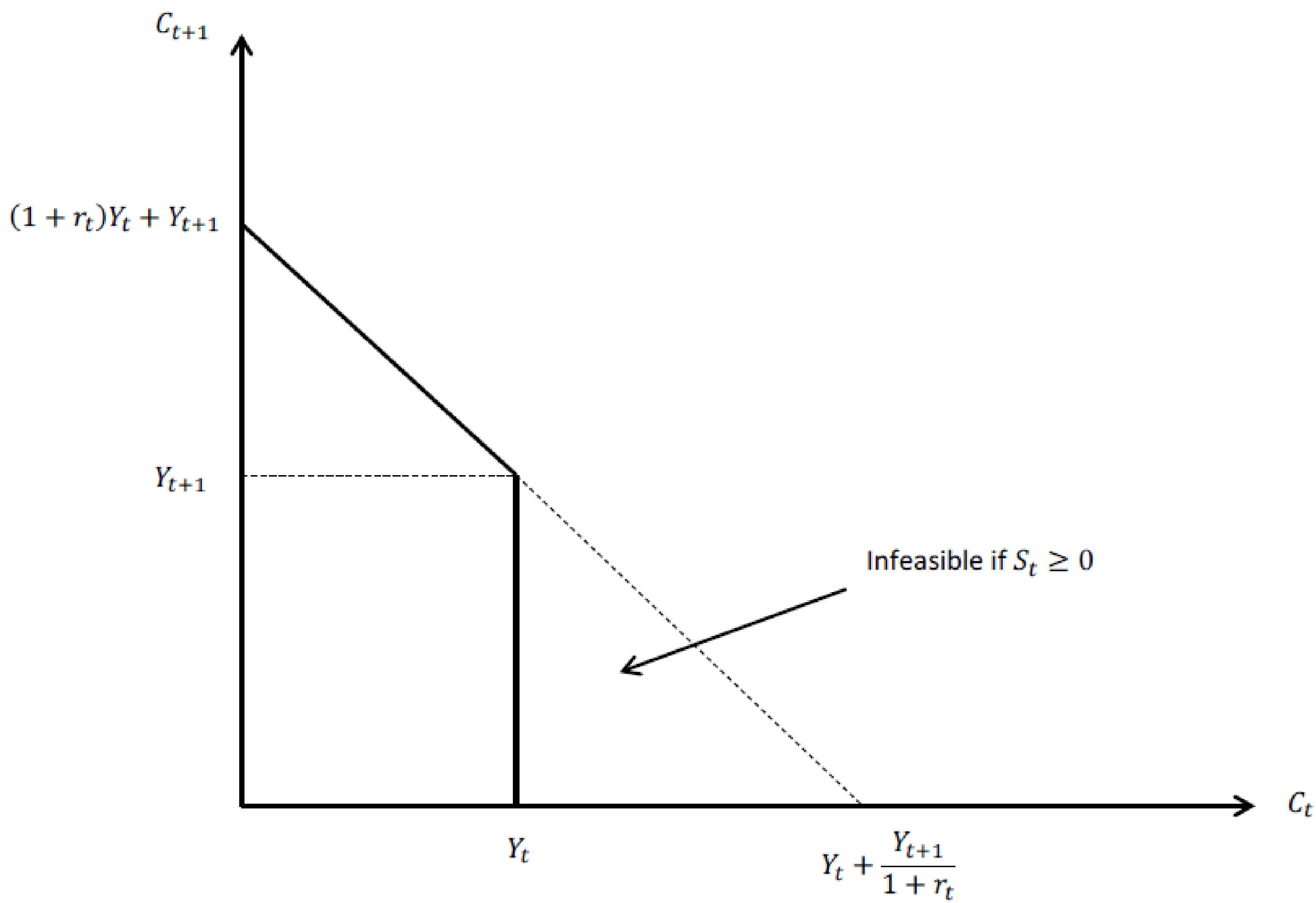


Figure 9.15: Borrowing Constraint: $r_t^b > r_t^s$

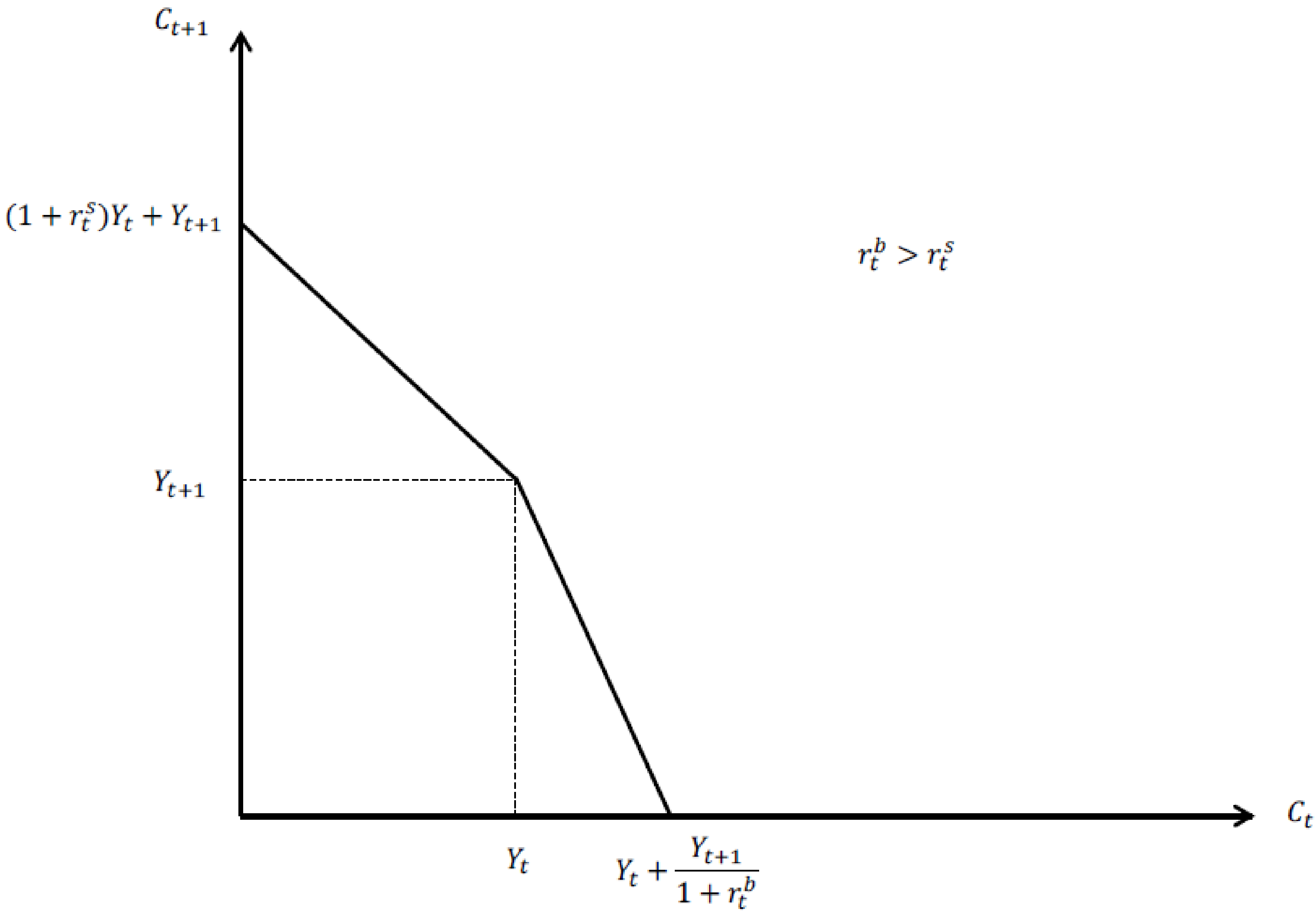


Figure 9.16: A Binding Borrowing Constraint

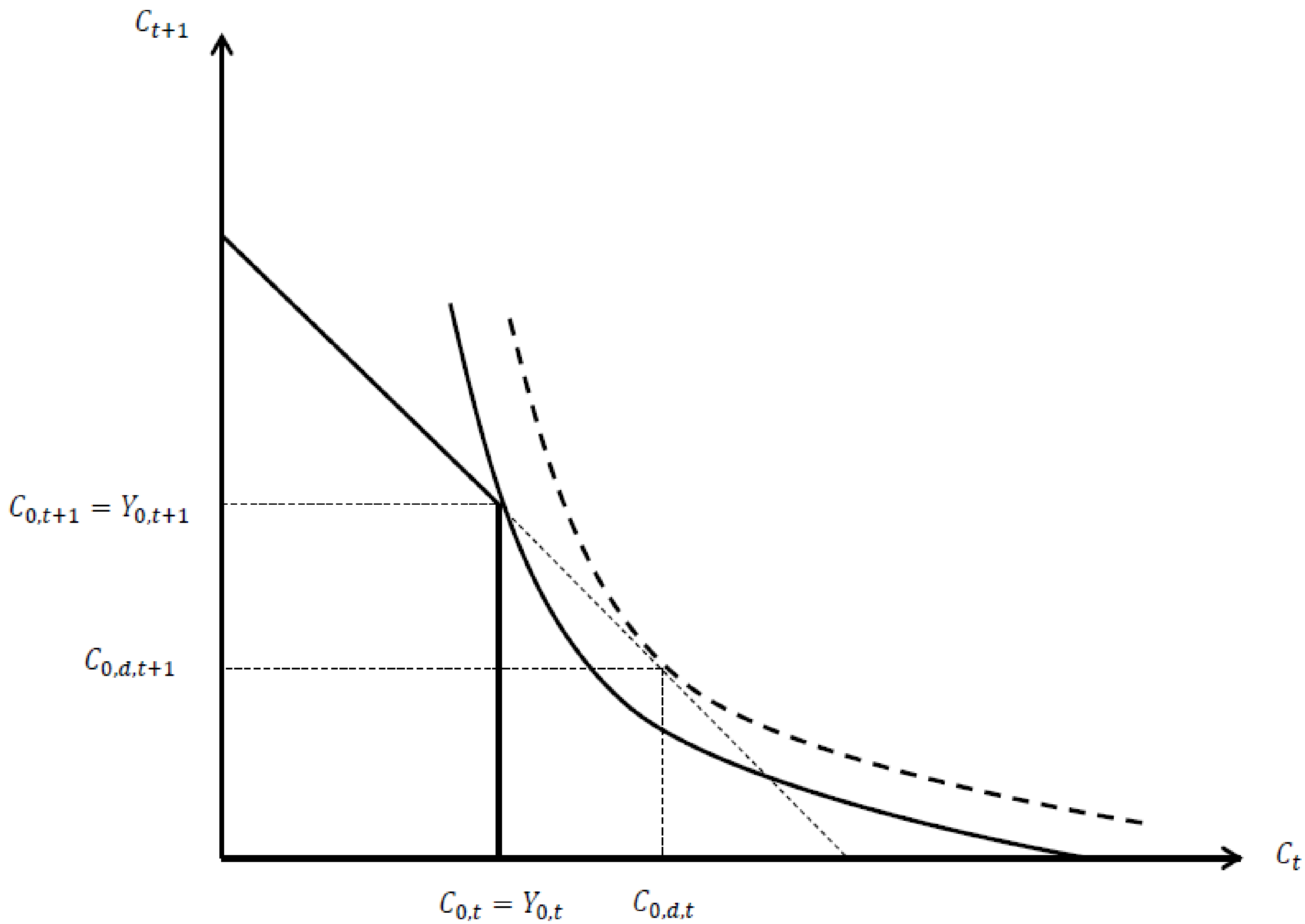


Figure 9.17: A Binding Borrowing Constraint, Increase in Y_t

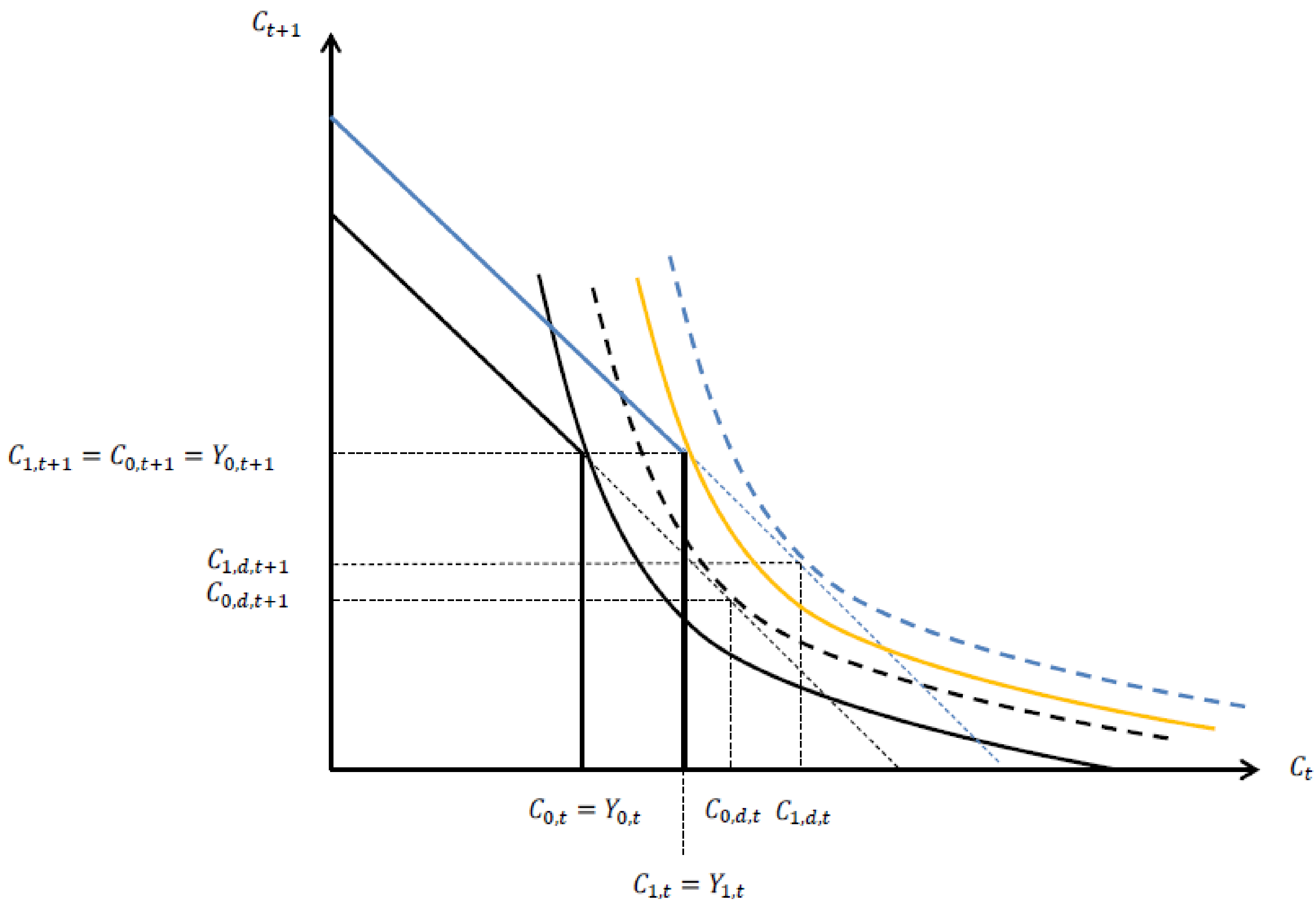


Figure 9.18: A Binding Borrowing Constraint, Increase in Y_{t+1}