Econ 702

Macroeconomics I

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Lecture 8: Optimal Growth Model

We saw in our model of optimizing households, if their utility function was logarithmic, they had an Euler equation of:

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + r_t\right),\ 0 < \beta < 1$$

In the Solow growth model, when a household saves in period t, the return to saving in period $t+1$ comes from rental of capital, at a rate R_{t+1}. So we have in this case, $r_t = R_{t+1}$. We can write:

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + R_{t+1}\right)$$

We have production and capital accumulation as in the Solow model. Assume a Cobb-Douglas production function. Assume $\delta = 0$ and $N_t = 1$, to keep things simple.

Then $R_t = \alpha A K_t^{\alpha-1}$, and

$$K_{t+1} - K_t = A_t K_t^{\alpha} - C_t$$

We can express the model in two equations:

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha A K_{t+1}^{\alpha-1}\right)$$

$$K_{t+1} - K_t = A_t K_t^{\alpha} - C_t$$

Steady State

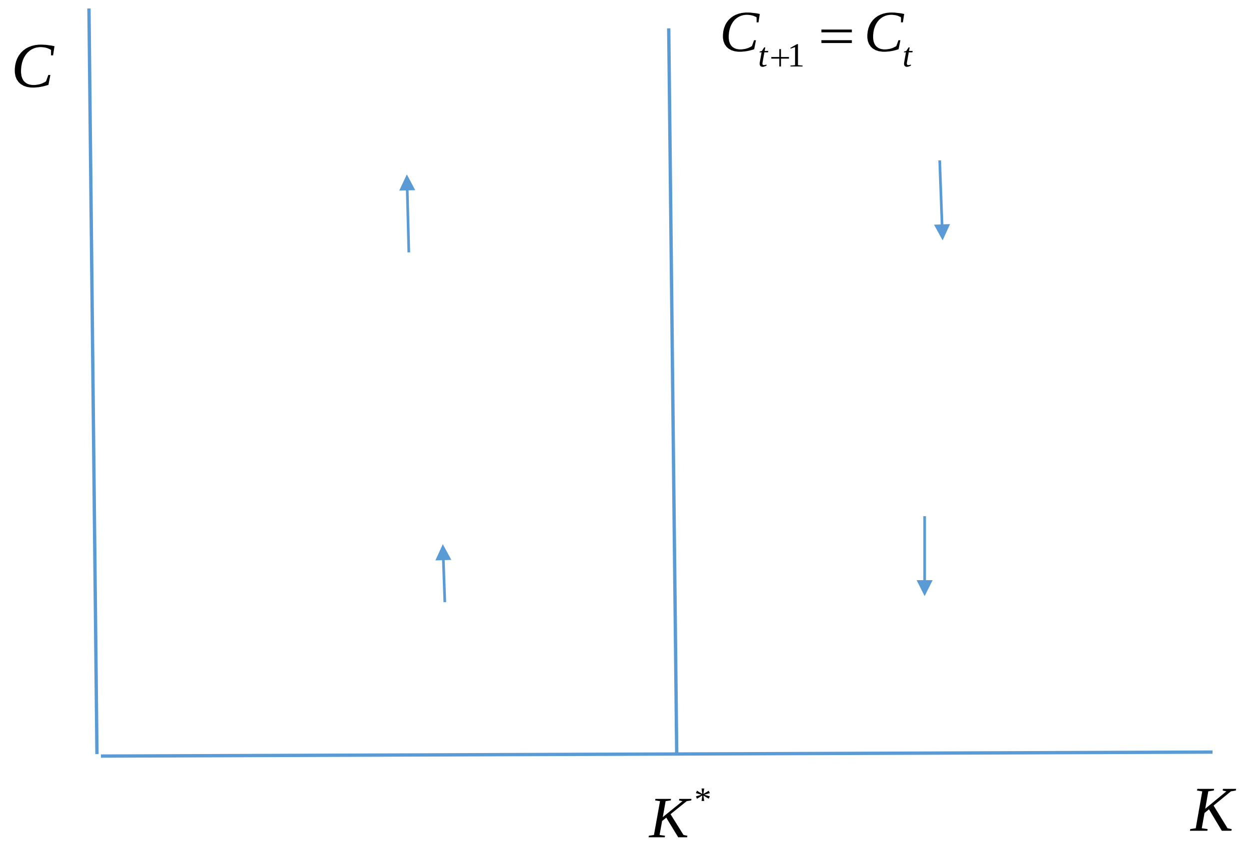
We have $\frac{C_{t+1}}{C_t} = 1$, so $1 = \beta\left(1 + \alpha A\left(K^*\right)^{\alpha-1}\right)$.

This tells us that $K^* = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$.

In the steady state, $C^* = Y^* = AK^{*\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$.

Higher productivity (A), greater "patience" (higher β), and a greater capital share (α), lead to higher consumption and income in the long run.

We can look at dynamics in a "phase diagram"



For points off the line where $K_t \neq K^*$, we must have $C_{t+1} \neq C_t$. For some levels of capital K_t, consumption is rising, $C_{t+1} > C_t$. For other values of K_t, consumption is falling, $C_{t+1} < C_t$.

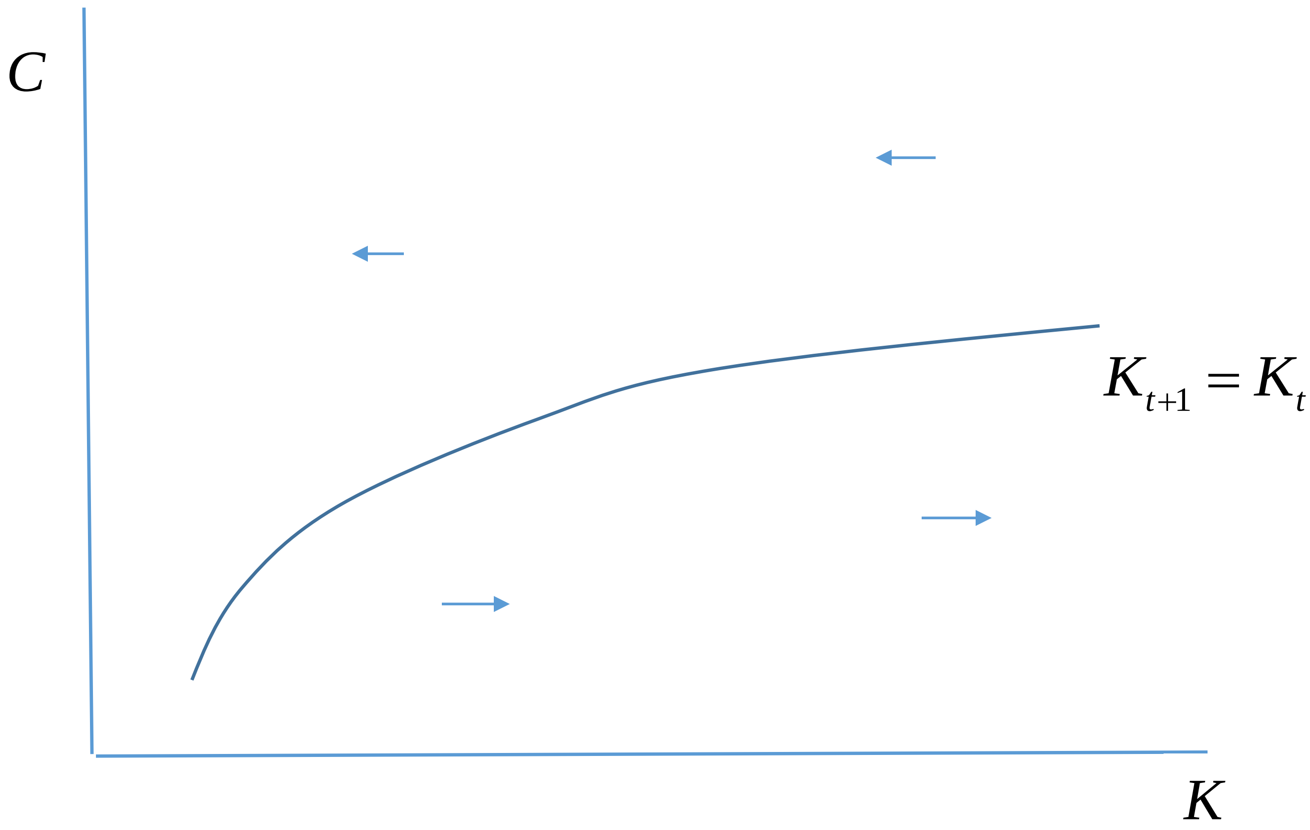
When consumption is rising, $C_{t+1} > C_t$, there is an upward pointing arrow to indicate that the variable on the vertical axis, C_t, is increasing.

The graph shows when $K_t < K^*$, consumption is increasing.
When $K_t < K^*$, the rental rate to capital is higher than R^*.

That is, if $K_t < K^*$, then $R_t = \alpha A K_t^{\alpha-1} > \alpha A K^{*\alpha-1} = R^*$. We can see that $\frac{dR_t}{dk_t} = (\alpha - 1)\alpha A K_t^{\alpha-2} < 0$ because $0 < \alpha < 1$. So when $K_t < K^*$, we have $\alpha A K_t^{\alpha-1} > \alpha A K^{*\alpha-1}$, which gives us $R_t > R^*$.

In turn, if $R_t > R^*$, then it must be the case that $\beta\left(1+R_t\right) > \beta\left(1+R^*\right)$. The Euler equation tells us $\frac{C_t}{C_{t-1}} = \beta\left(1+R^*\right)$. When $\beta\left(1+R^*\right)=1$, consumption does not grow, $\frac{C_t}{C_{t-1}} = 1$. It follows that if $\beta\left(1+R_t\right) > \beta\left(1+R^*\right)$, we must have $\frac{C_t}{C_{t-1}} > 1$. Consumption is growing.

Similarly, when $K_t > K^*$, consumption is falling.



C
$K_{t+1} = K_t$
K

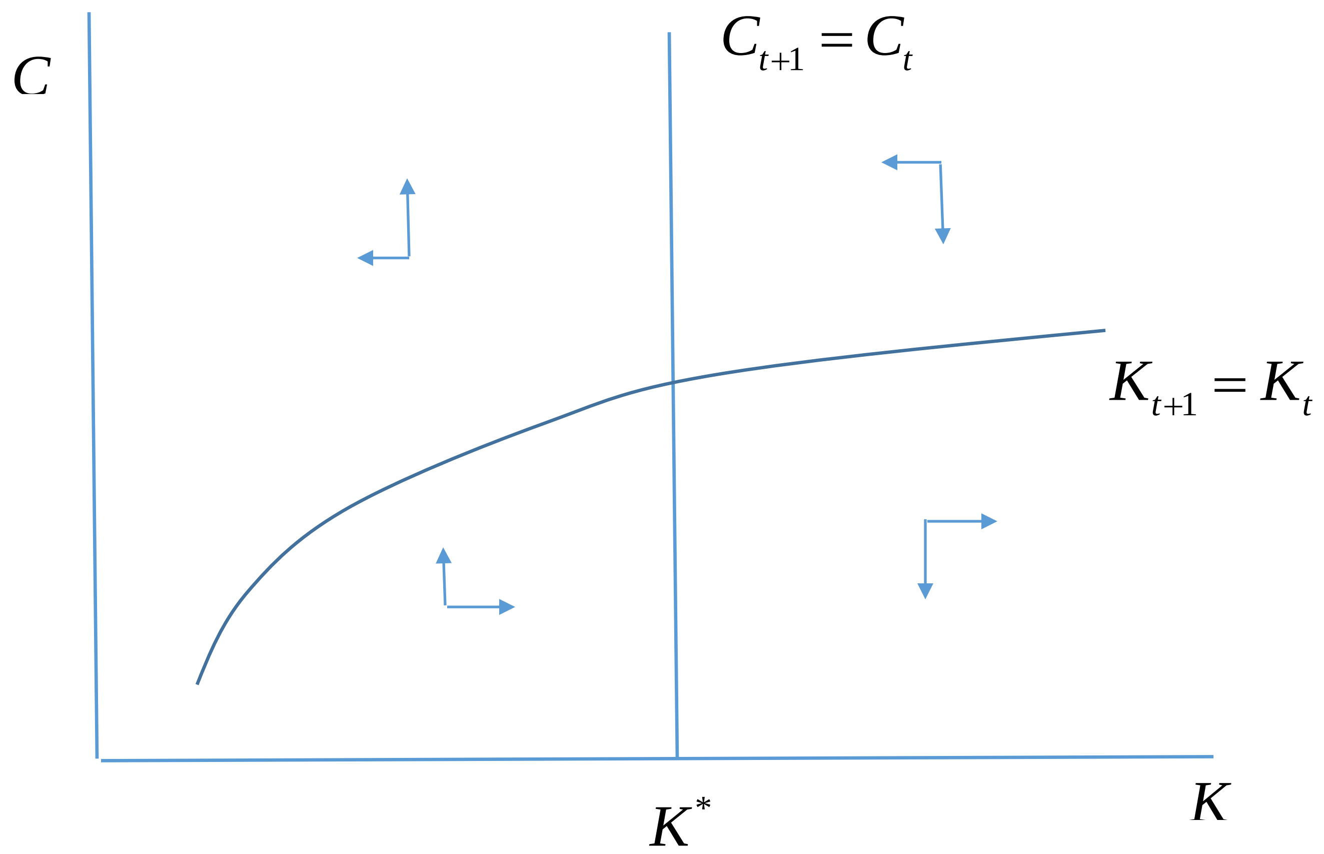
Points above the curve $C_t = AK_t^\alpha$ are points where $C_t > AK_t^\alpha$.

This says consumption is greater than output, so the capital stock must be falling.

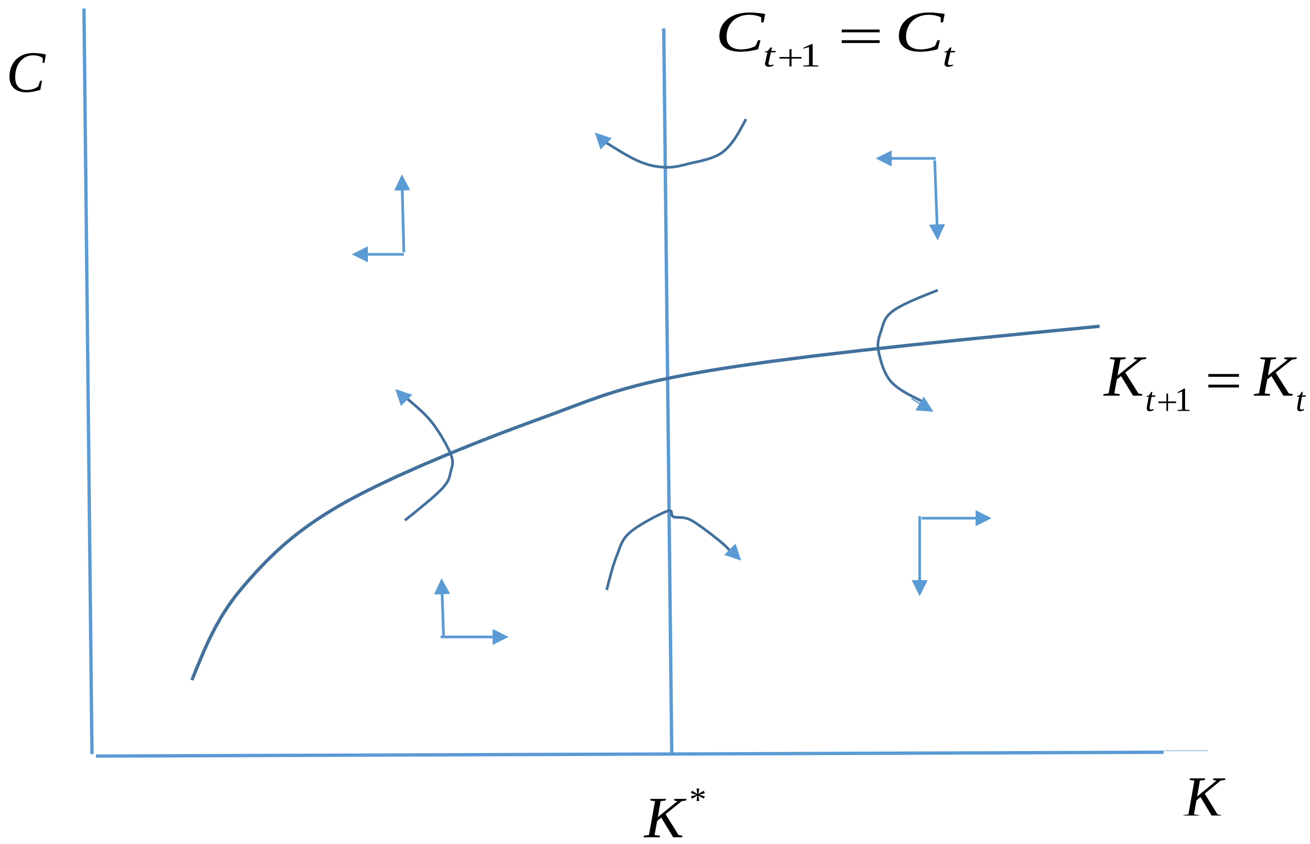
We have $K_{t+1} - K_t = AK_t^\alpha - C_t$, so if $C_t > AK_t^\alpha$, then we must have $K_{t+1} - K_t < 0$.

This is indicated on the graph by leftward pointing arrows in the region in which $C_t > AK_t^\alpha$. These arrows indicate that the variable on the horizontal axis, K_t, is falling in this region.

In the region in which $C_t < AK_t^\alpha$, capital is rising, indicated by rightward pointing arrows.

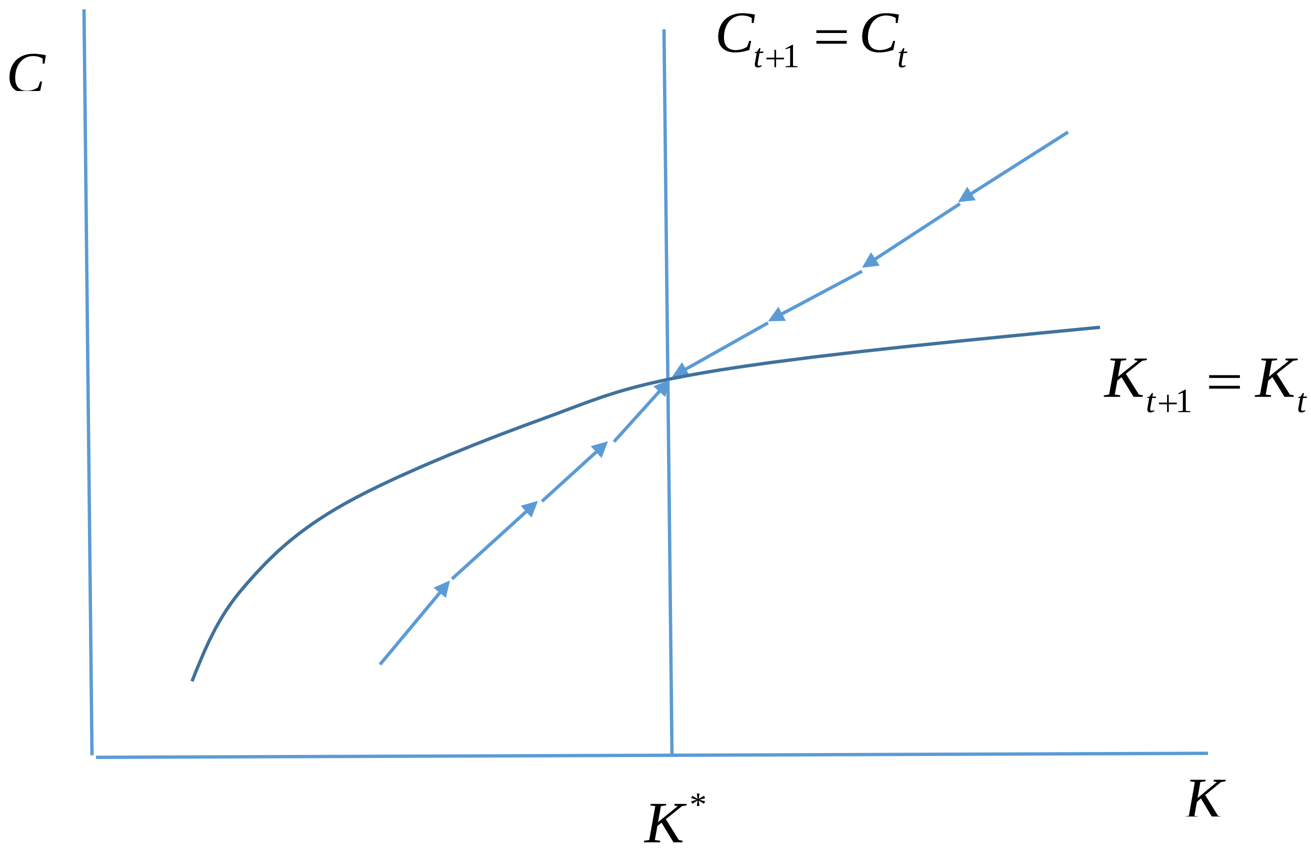


C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K^*
K



C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K^*
K

The “saddle path”:



If this were a two-period model, the budget constraint would be:

$$C_t + \frac{1}{1+R_{t+1}} C_{t+1} = Y_t + \frac{1}{1+R_{t+1}} Y_{t+1}$$

In a three-period model, we would have:

$$C_t + \frac{1}{1+R_{t+1}} C_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} C_{t+2} = Y_t + \frac{1}{1+R_{t+1}} Y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} Y_{t+2}$$

Ours is an infinite-horizon model. The budget constraint is:

$$C_t + \frac{1}{1+R_{t+1}} C_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} C_{t+2} + \ldots$$

$$= Y_t + \frac{1}{1+R_{t+1}} Y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} Y_{t+2} + \ldots$$

The only dynamic path that satisfies

$$K_{t+1} - K_t = AK_t^{\alpha} - C_t$$

$$\frac{C_{t+1}}{C_t} = \beta\left(1 + \alpha AK_{t+1}^{\alpha-1}\right)$$

$$C_t + \frac{1}{1+R_{t+1}}C_{t+1} + \frac{1}{\left(1+R_{t+1}\right)\left(1+R_{t+2}\right)}C_{t+2} + \ldots$$
$$= Y_t + \frac{1}{1+R_{t+1}}Y_{t+1} + \frac{1}{\left(1+R_{t+1}\right)\left(1+R_{t+2}\right)}Y_{t+2} + \ldots$$

is the saddle path

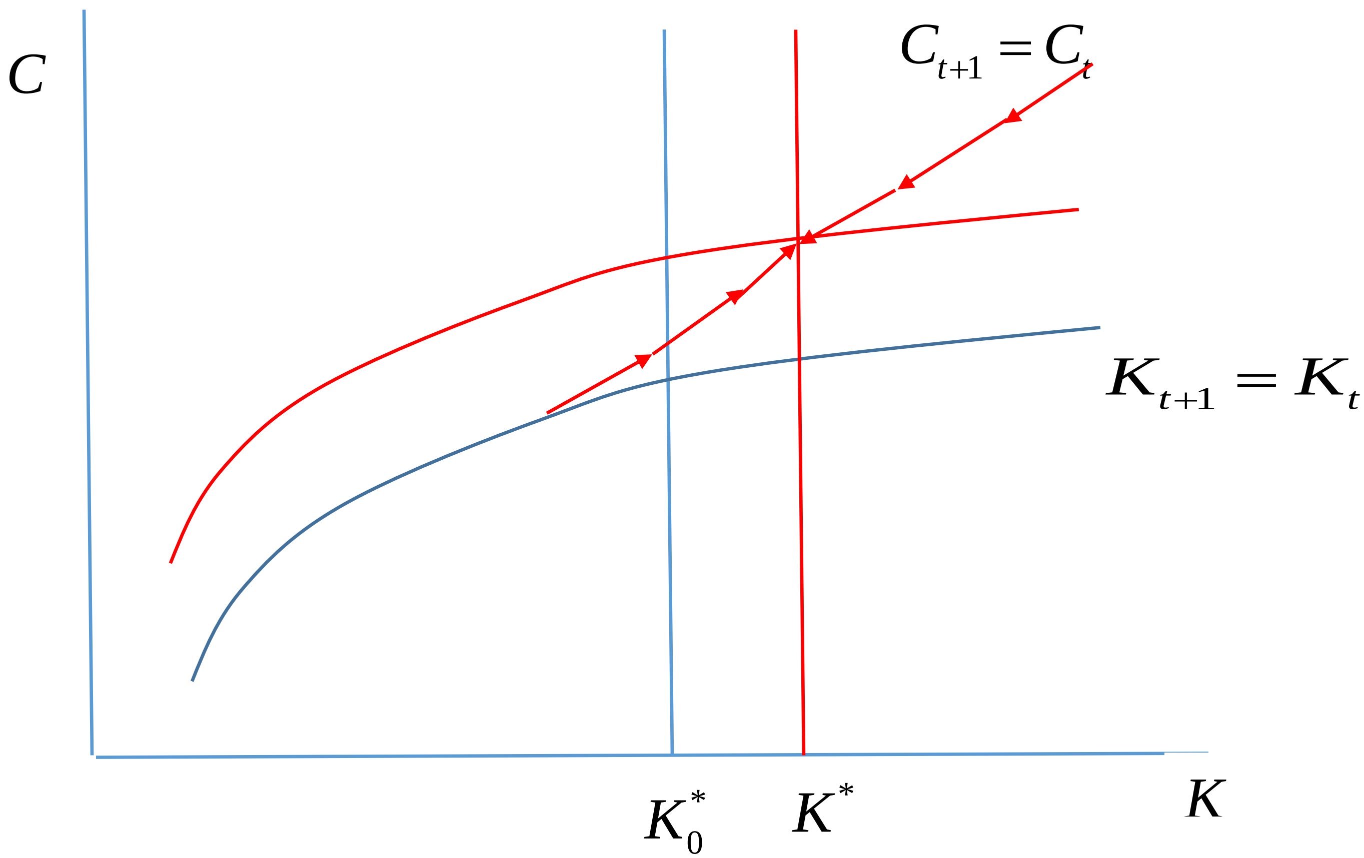
An increase in A

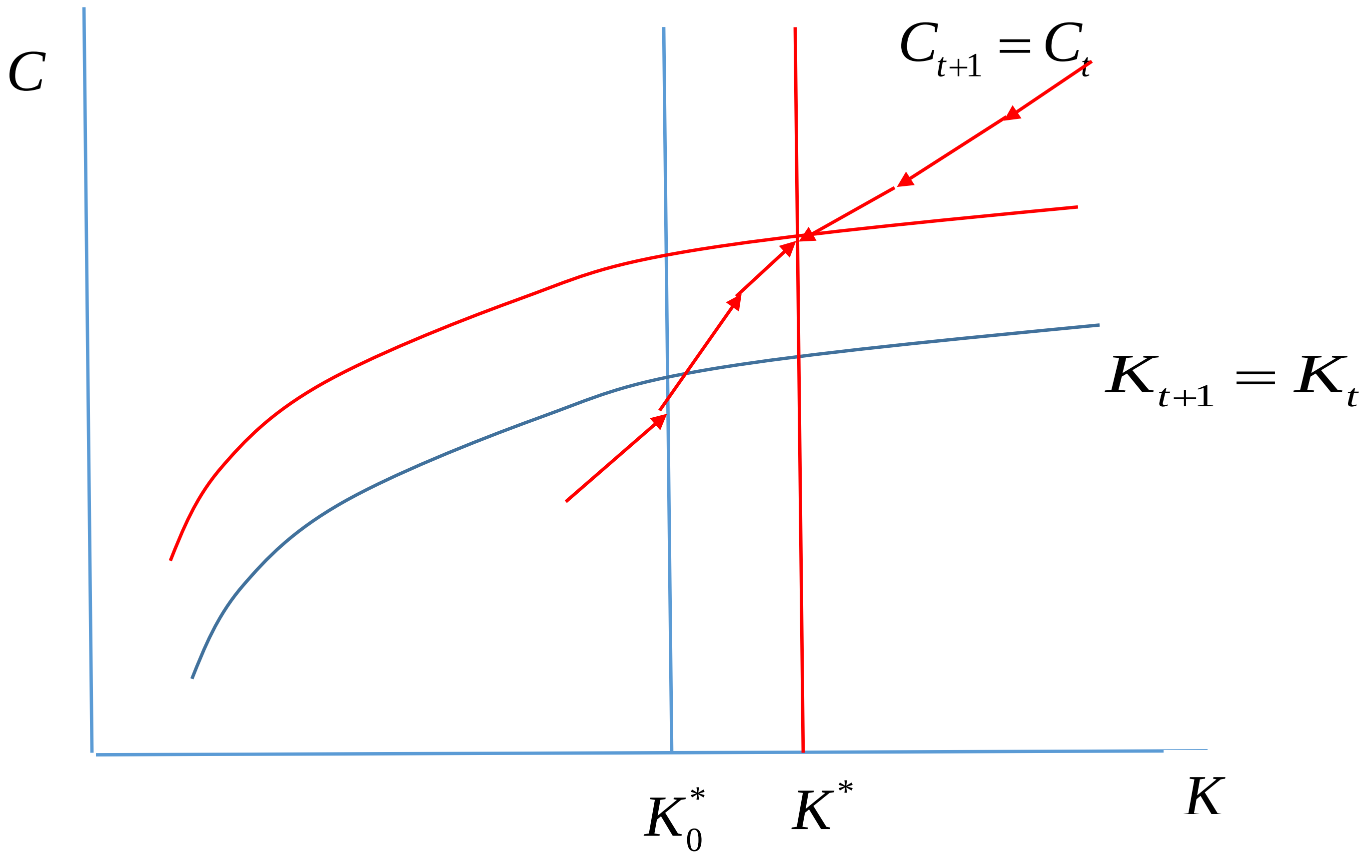
The equation of the $\frac{C_t}{C_{t-1}} = 1$ is $K_t = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$. An increase in A shifts that line to the right.

The equation of the $K_{t+1} = K_t$ line is $C_t = AK_t^{\alpha}$. An increase in A shifts that line up.

The graph below shows an immediate increase in C_t, then K_t rises toward the new steady state. Some output is consumed, some is saved. As output rises, consumption rises, and saving also rises – until the steady state.

Note that in contrast to the Solow model, C_t could fall initially, for example if β is large enough.


C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K_0^*
K^*
K



C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K_0^*
K^*
K

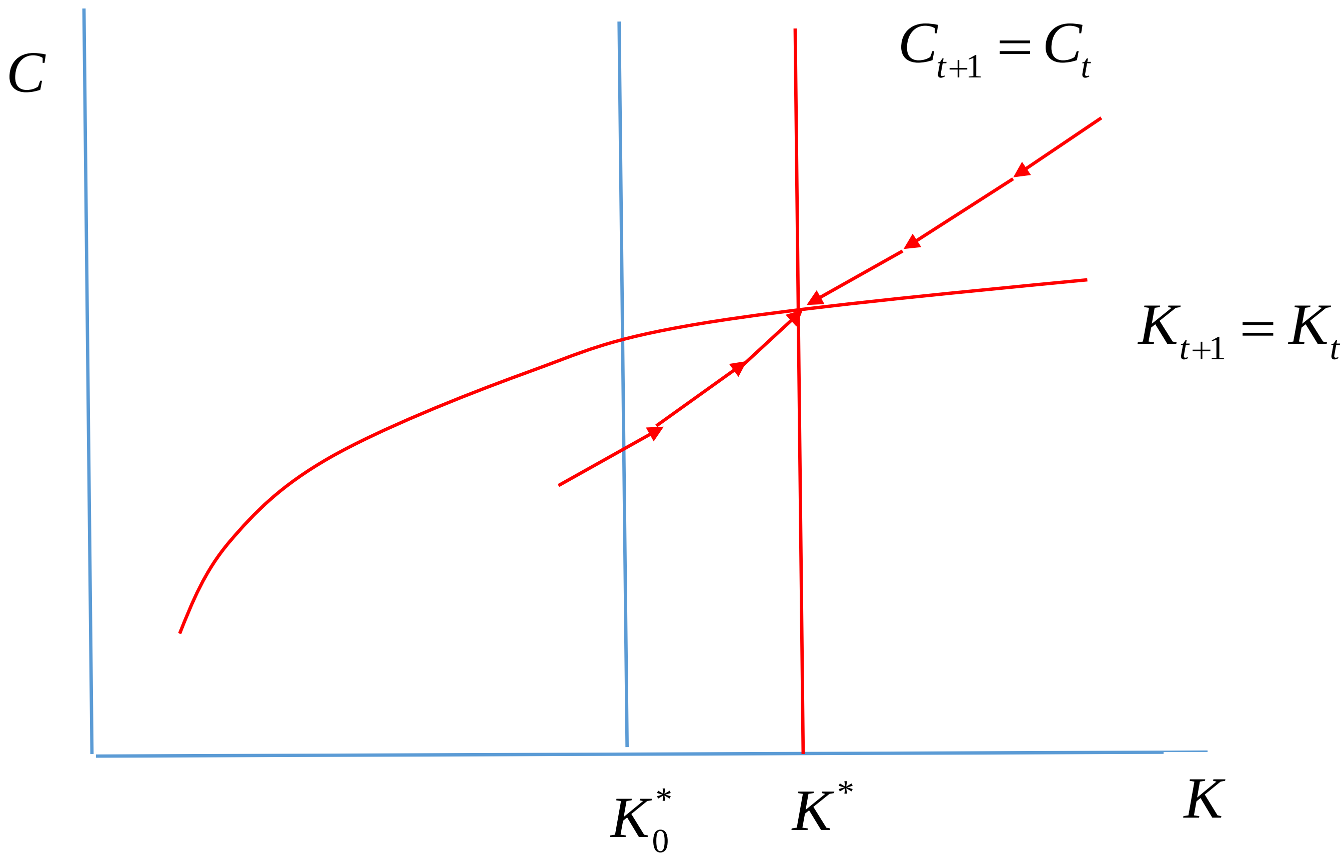
An increase in β

The equation of the $\frac{C_t}{C_{t-1}} = 1$ is $K_t = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$. An increase in β shifts that line to the right.

The equation of the $K_{t+1} = K_t$ line is $C_t = AK_t^{\alpha}$. An increase in β does not shift that line.

Consumption falls initially as saving rises.

Then K_t rises toward the new steady state. Some output is consumed, some is saved. As output rises, consumption and saving also rise – until the steady state.



C
$C_{t+1} = C_t$
$K_{t+1} = K_t$
K_0^*
K^*
K

Lessons from this model:

We still have a steady state with no long run growth!

Saving more this period, increases the capital stock next period.

As the capital stock increases, the marginal product of capital falls.

Output approaches a steady state.

The optimal consumption path when below the steady state is to have high consumption growth initially. But as the MPK falls, consumption growth falls, and consumption growth approaches zero as R^* goes toward $\frac{1-\beta}{\beta}$.

Consumption path is always optimal (no “golden rule”).