Notes on the growth model with optimal consumption.

In Chapter 9, we derived the consumption Euler equation. (See equation 9.22). There we found that the marginal rate of substitution between consumption at time *t* and consumption at time $t+1$ is equal to the gross rate of return, $1+r_t$, from saving at time *t*:

$$(1) \qquad \frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_t .$$

That equation was derived for a person that was living only two periods, but the intuition of that condition tells us that it would hold more generally for someone that lives a much longer life. In this model, we will assume that the consumer is infinitely-lived. Obviously that is not a realistic assumption, but it is made for convenience. Equation (1) says that if we give up one unit of consumption in period *t* in order to get one unit of consumption in period $t+1$, the ratio of the marginal utilities of period *t* and $t+1$ should equal the gross return on consumption. When a person gives up a unit of consumption at time *t*, they lose $u'(c_t)$ in marginal utility. Each unit of marginal utility that they gain in period $t+1$ is worth $\beta u'(c_{t+1})$ units of marginal utility today. Since the saver earns $1+r_t$ on each unit saved, the marginal utility gain from saving one unit is $(1+r_t)\beta u'(c_{t+1})$. The consumer saves additional units of consumption at time *t* until the marginal utility loss of the next unit of consumption given up just equals the marginal utility gain from increasing consumption at $t+1$.

Keep in mind that $0 < \beta < 1$. β is the "utility discount factor", the weight that the consumer puts on utility of consumption in period $t+1$ relative to utility in period *t*. The consumer prefers consumption sooner rather than later, which is why we assume $\beta < 1$. For example, thinking about it at time t, the utility that the consumer gets from an ice cream cone postponed for three months (which may be one time period in our model) might be only 0.99 times the utility that she would feel if she could have the ice cream today.

In these notes, we will assume that utility is logarithmic: $u(c_t) = \ln(c_t)$. Then we can write the condition (1) as:

$$(2) \qquad \frac{c_{t+1}}{c_t} = \beta(1+r_t)$$

We will assume that equation (2) determines the consumption/saving decision in the optimizing version of the Solow growth model. That is, this equation replaces the equation we had before, which stated that consumption was a constant fraction of income: $c_t = (1-s)y_t$. By saying that equation is replaced, we mean it is not an equation of the model.

In the Solow growth model, saving takes place in the form of capital accumulation. Households own capital which they rent out to firms. If a household saves a unit of capital at time *t*, the capital is put to productive use at time $t+1$. At $t+1$, the household can rent that unit of capital to the firm, and earn a rent equal to R_{t+1}. The return to saving at time *t* comes from the rental of the capital at time $t+1$. So, in the Solow model, R_{t+1} is the return to saving – it is what we have called r_t above. We can write this version of equation (2):

$$(3) \qquad \frac{c_{t+1}}{c_t} = \beta\left(1+R_{t+1}\right).$$

Otherwise, let's take the rest of the Solow model as before, with the Cobb-Douglas production function. We derived the dynamic equation for accumulation of capital per worker, which was given by:

$$(4) \qquad K_{t+1} = AK_t^{\alpha}N_t^{1+\alpha} - c_t + \left(1-\delta\right)K_t.$$

Recall that the equation says that capital in period $t+1$ is equal to the amount of new saving in period *t*, plus the undepreciated capital carried over from period *t*. New saving is equal to output minus consumption: $AK_t^{\alpha}N_t^{1+\alpha} - c_t$. As before, let k_t be capital per worker, but let's assume the number of workers is held constant, and measure the number in units such that $N_t = 1$. That means $K_t = k_t$. Also, to keep the model relatively simple, assume that capital does not depreciate: $\delta = 0$.

We can write the capital accumulation equation (4) as:

$$(5) \qquad k_{t+1} - k_t = Ak_t^{\alpha} - c_t$$

Then, as we derived in our chapter on the basic Solow model,

$$(6) \qquad R_t = \alpha Ak_t^{\alpha-1}.$$

That allows us to write equation (3) as:

$$(7) \qquad \frac{c_{t+1}}{c_t} = \beta\left(1+\alpha Ak_{t+1}^{\alpha-1}\right).$$

Equations (5) and (7) are the dynamic equations of the model that, together, show how consumption and capital evolve over time. Unlike the basic Solow model, we cannot collapse the dynamics of consumption and capital accumulation into a single equation. In the basic Solow model, consumption depended only on current output per worker, which in turn depended only on current capital per worker, so we could substitute out c_t in equation (5) as a function of k_t and arrive at a dynamic equation for k_{t+1} and k_t. Instead, in this model, the growth rate of consumption, which relates c_{t+1} to c_t, depends on the return to capital at time $t+1$, which in turn depends on capital per worker at time $t+1$. We need both equations (5) and (7) to analyze the dynamics.

Steady State

This model has a steady state in which consumption and capital per worker are constant. For consumption, being in the steady state means $\frac{c_{t+1}}{c_t}=1$. Then, imposing this condition onto equation (7), we get:

(8) $$1=\beta\left(1+\alpha A\left(k^*\right)^{\alpha-1}\right).$$

k^* is the steady-state capital per worker. Equation (8) can be solved to find:

(9) $$k^*=\left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}.$$

Equation (9) is an important equation because it tells us that the long-run capital per worker in the optimizing Solow model depends positively on three parameters: total factor productivity, *A*; the utility discount factor, β; and, capital's weight α in the Cobb-Douglas production function. As we can see from equation (6), higher total factor productivity or a greater share for capital (α) increase the return to saving. Households have a greater incentive to save because the rental rate on capital is higher, the higher are *A* and α.

For any given rental rate, households that are more patient will save more. "Patient" households are ones that put more weight on utility of future consumption. They have a higher β. From the Euler equation (7), we see that more patient households will put off consumption today (lower c_t) in order to get more consumption in the future (higher c_{t+1}.)

Higher capital per worker in the steady state will imply higher output per worker and higher consumption in the steady state. This can be seen by setting $k_{t+1}-k_t=0$ in equation (5). We find:

(10) $$y^* = Ak^{*\alpha} = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$$

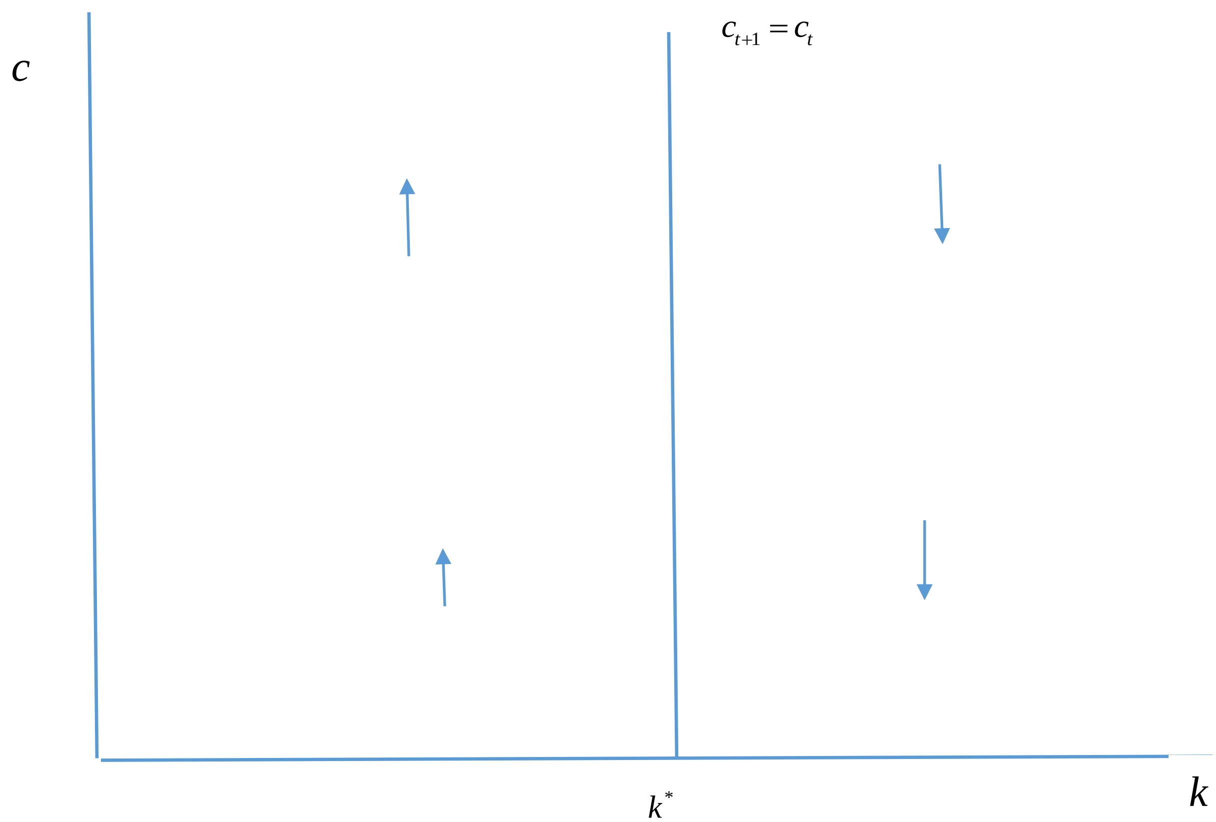
(11) $$c^* = y^* = A^{\frac{1}{1-\alpha}}\left(\frac{\beta}{1-\beta}\alpha\right)^{\frac{\alpha}{1-\alpha}}$$

Higher productivity (A), greater "patience" (higher β), and a greater capital share (α), lead to higher consumption and income in the long run.

These equations characterize the steady state of the model, but do not tell us anything about the dynamic paths of consumption, capital or output. We turn to that next.

<u>Dynamics</u>

We can look at dynamics in a "phase diagram". Let's first consider consumption dynamics. We would like to understand how consumption evolves at a point in time *t*, given the capital stock that we enter into period *t* with.



We know that when $k_t = k^*$, that consumption is not changing, $c_{t+1} = c_t$. That conclusion follows from equation (7) and our analysis of the steady state. On the graph above, the set of points where consumption is not changing over time is graphed by the vertical line rising from the point where $k_t = k^*$. That line is labeled $c_{t+1} = c_t$, indicating that along that line, consumption does not change over time.

For points off that line, where $k_t \neq k^*$, it follows logically that consumption is changing, so $c_{t+1} \neq c_t$. For some levels of capital k_t, we'll find that consumption is rising, $c_{t+1} > c_t$. For other values of k_t, we will find that consumption is falling, $c_{t+1} < c_t$.

The graph above indicates the region in which consumption is rising, $c_{t+1} > c_t$, with an upward pointing arrow to indicate that the variable on the vertical axis, c_t, is increasing in that region. Likewise, the region in which consumption is falling, $c_{t+1} < c_t$, is marked with a downward pointing arrow to indicate that the variable on the vertical axis, c_t, is decreasing in that region.

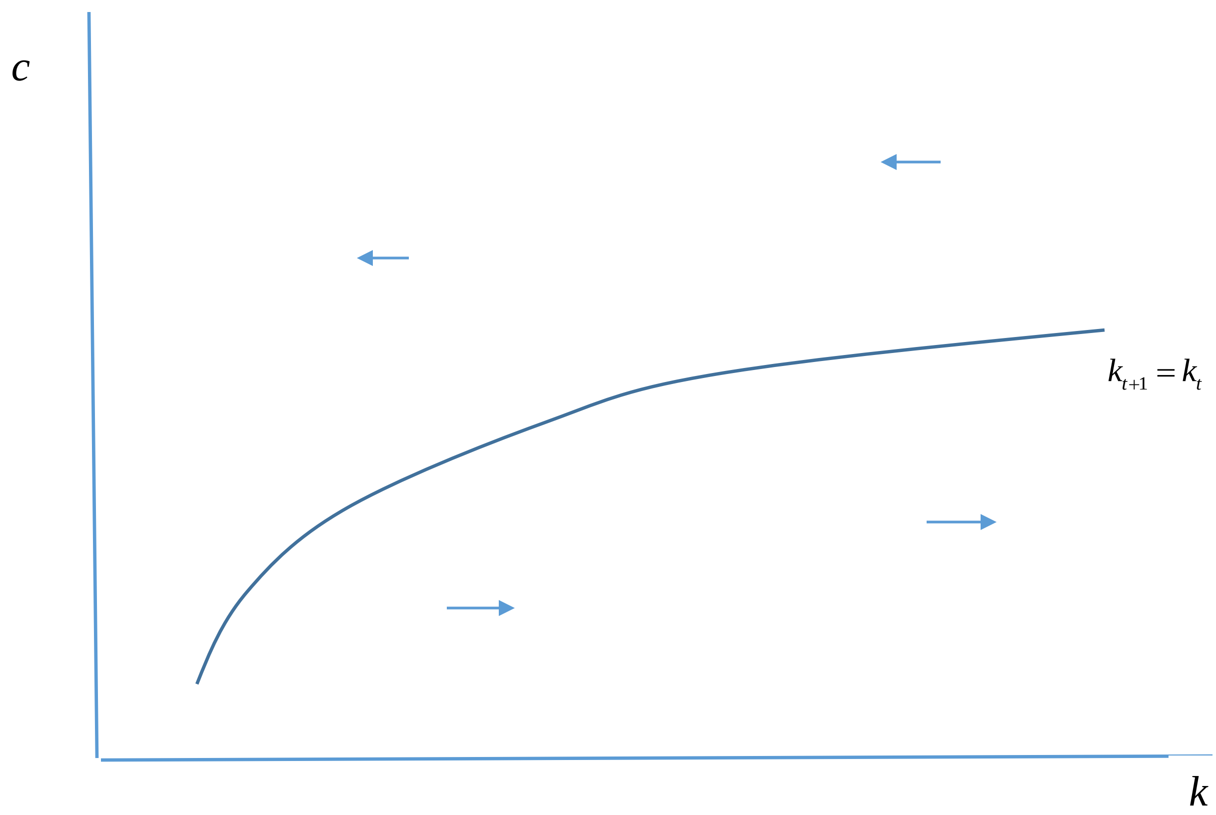
The graph indicates that when $k_t < k^*$, consumption is increasing. Why is that the case? When $k_t < k^*$, we have that the rental rate to capital is higher than R^*. That is, if $k_t < k^*$, then $R_t = \alpha A k_t^{\alpha-1} > \alpha A k^{*\alpha-1} = R^*$. This follows because of diminishing marginal productivity. That is, we can see that $\frac{dR_t}{dk_t} = (\alpha - 1)\alpha A k_t^{\alpha-2} < 0$ because $0 < \alpha < 1$. So when $k_t < k^*$, we have $\alpha A k_t^{\alpha-1} > \alpha A k^{*\alpha-1}$, which gives us $R_t > R^*$. In simple words, when there is less capital, the return to capital is higher.

In turn, if $R_t > R^*$, then it must be the case that $\beta(1+R_t) > \beta(1+R^*)$. Equation (3) tells us that $\frac{C_t}{C_{t-1}} = \beta(1+R^*)$. When $\beta(1+R^*) = 1$, consumption does not grow, $\frac{C_t}{C_{t-1}} = 1$. It follows that if $\beta(1+R_t) > \beta(1+R^*)$, we must have $\frac{C_t}{C_{t-1}} > 1$. Consumption is growing.

The logic tells us when the capital per worker is below the steady state level, the rental rate of capital is higher than in the steady state because of diminishing marginal product of capital. In this case, households have an incentive to save, and consumption will be rising over time.

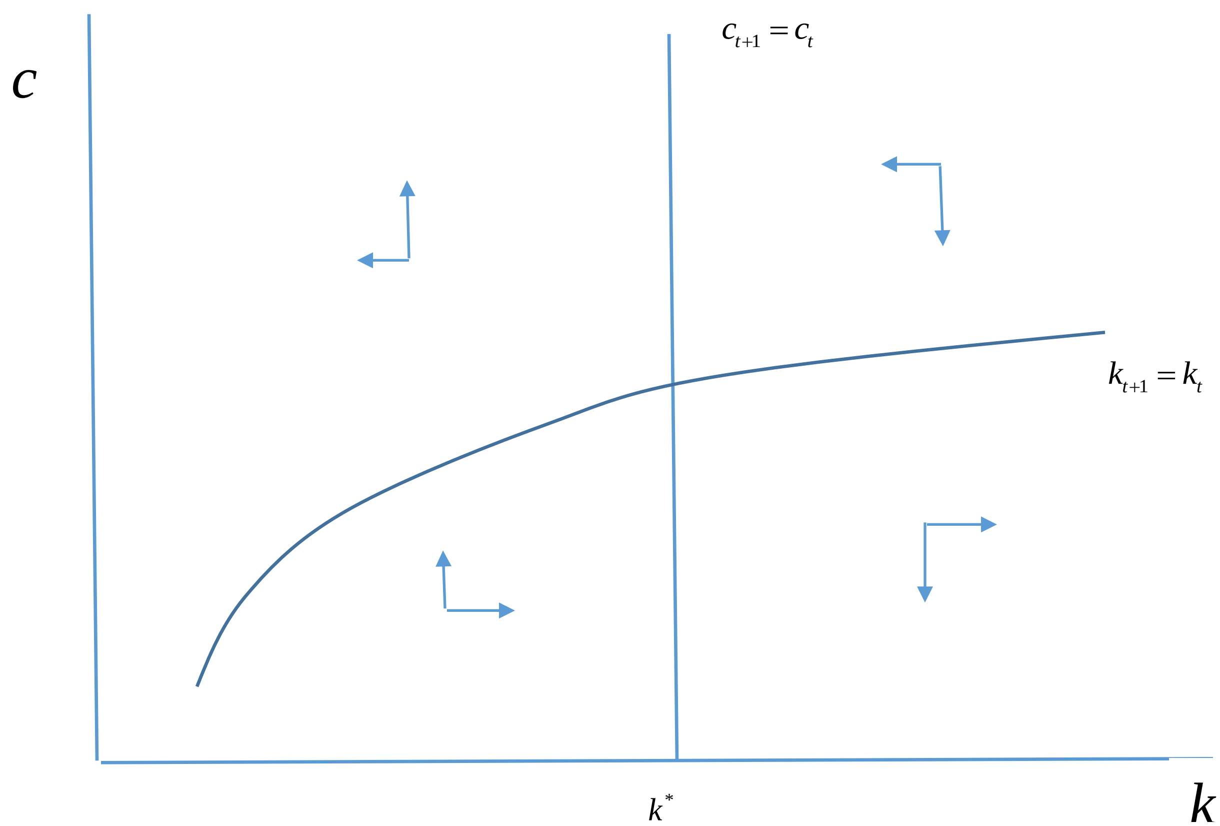
Similarly, when $k_t > k^*$, consumption is falling.

The next graph shows us how capital changes over time. From equation (5), when consumption equals output, $c_t = Ak_t^{\alpha}$, capital is constant, $k_{t+1} = k_t$. The line $c_t = Ak_t^{\alpha}$ is graphed below, and labeled $k_{t+1} = k_t$. Along that line, the capital stock is constant. For a given level of capital, if $c_t \neq Ak_t^{\alpha}$, then the capital stock is not constant, $k_{t+1} \neq k_t$. That is, for points on the graph that are not along the curve where $c_t = Ak_t^{\alpha}$, the capital stock must be changing over time – either rising or falling.



In the graph, points that lie above the curve $c_t = Ak_t^\alpha$ are points where $c_t > Ak_t^\alpha$. But if $c_t > Ak_t^\alpha$, then consumption is greater than output, so the capital stock must be falling. That is, from equation (5), $k_{t+1} - k_t = Ak_t^\alpha - c_t$, so if $c_t > Ak_t^\alpha$, then we must have $k_{t+1} - k_t < 0$. This is indicated on the graph by leftward pointing arrows in the region in which $c_t > Ak_t^\alpha$. These arrows indicate that the variable on the horizontal axis, k_t, is falling in this region. Similarly, in the region in which $c_t < Ak_t^\alpha$, capital is rising, indicated by rightward pointing arrows.

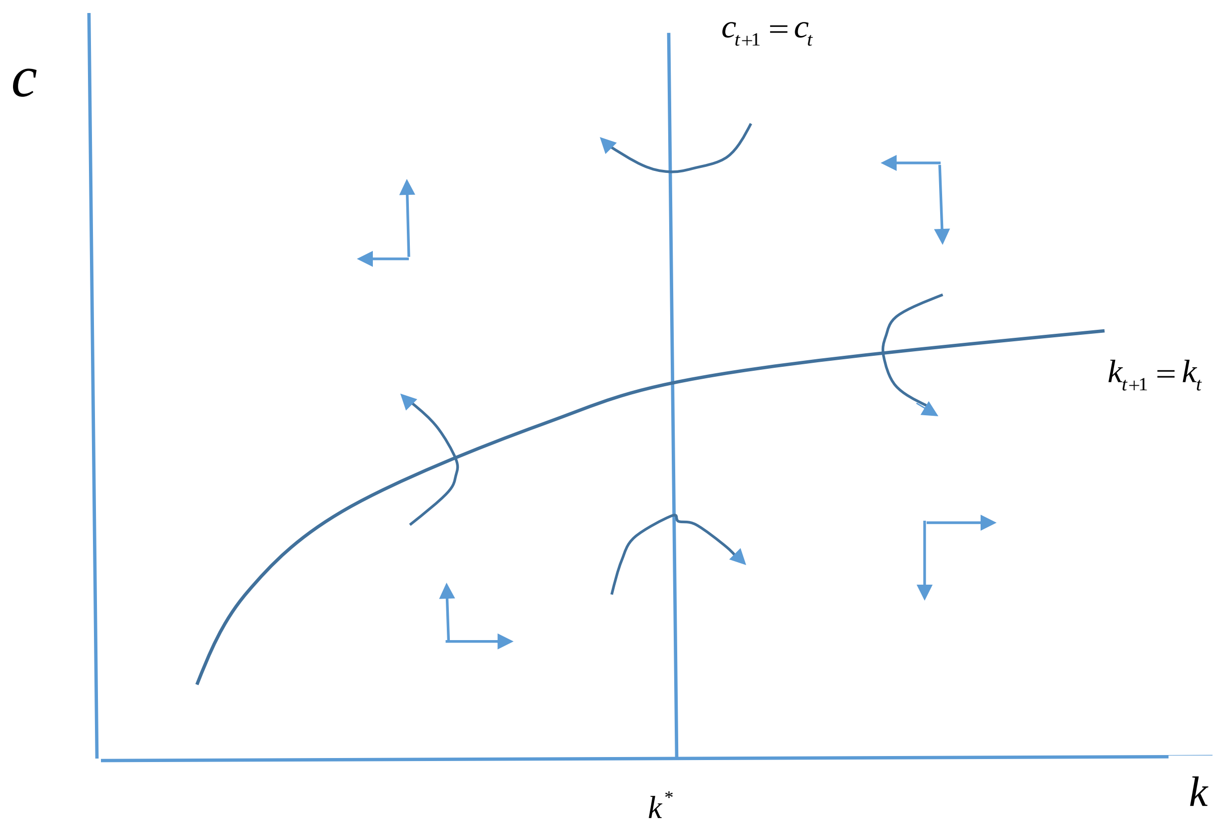
The next graph puts the previous two graphs together to indicate the direction of motion for consumption and capital.



The graph divides the space into four quadrants. The arrows in each quadrant show the direction of motion of consumption and the capital stock in that quadrant.

Take, for example, the upper-left-hand quadrant. In that area, consumption is rising, and the capital stock is falling. Clearly, in that region, both consumption and capital are always moving away from the steady state. There is no way to reverse direction and head toward the steady state if c_t and k_t are in that region. The same is true for the lower-right-hand quadrant.

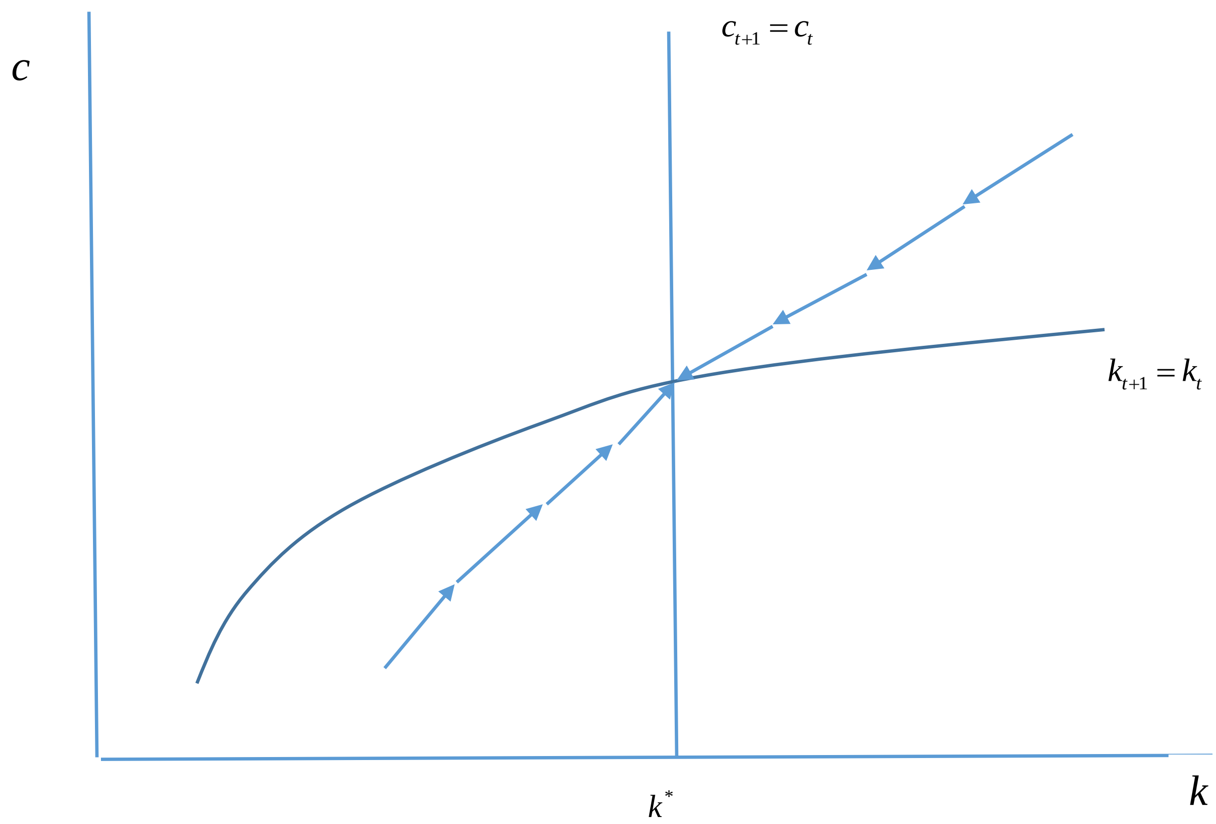
More interesting are the upper-right-hand and lower-left-hand quadrants.



As we can see in the graph above, from the upper-right quadrant, there are a few possibilities. In that region, capital is falling and consumption is falling. It is possible that the path of consumption leads the economy into the upper-left quadrant, where capital rises and consumption falls, moving away from the steady state. Such a path is indicated by the curved arrow that crosses the $c_{t+1} = c_t$ line from the upper-right to the upper-left quadrant. Another possibility is that consumption and capital fall and the economy enters the lower-right quadrant. One such path is indicated by the curved arrow that crosses the $k_{t+1} = k_t$ from the upper-right to the lower-right quadrants.

Similarly, if the economy is in the lower-left quadrant, it might transit to the upper-left or lower-right quadrants.

But there is also exactly one path in the upper right quadrant, and one in the lower-left quadrant, that bring capital and consumption to the steady state as indicated here:



Mathematically, there are an infinite number of paths for consumption and capital that satisfy the Euler equation for consumption and the capital accumulation equation. Those paths are all indicated in the phase diagram. But there is only one path that leads to the steady state. (Mathematically, this path is called the "saddle path".)

The saddle path is the only path that satisfies equations (5) and (7) and also satisfied the lifetime budget constraint. The idea of the lifetime budget constraint was introduced in chapter 9 (see equation 9.17). When the lifetime budget constraint holds with equality, that equation in the two-period model says:

$$c_t + \frac{1}{1+r_t} c_{t+1} = y_t + \frac{1}{1+r_t} y_{t+1} .$$

The left-hand-side of that equation is the present value of lifetime consumption, and the right-hand-side is the present value of lifetime income. Chapter 9 discusses why we consider the case in which the lifetime budget constraint holds with equality. The present value of consumption cannot exceed the present value of income, because that would imply the person would die as a debtor and never pay back his debts. But the present value of consumption cannot be less than the present value of income, because the person would die without consuming all of his income, which would not be rational in this model.

In our case, since we have $R_{t+1} = r_t$, if our model were a two-period model, we would write the lifetime budget constraint as:

$$c_t + \frac{1}{1+R_{t+1}} c_{t+1} = y_t + \frac{1}{1+R_{t+1}} y_{t+1} .$$

But our model goes beyond two periods. What if we had a three-period model? How do we discount income earned in period y_{t+2}? In period $t+1$, the present value of income earned in period $t+2$ is equal to $\frac{1}{1+R_{t+2}} y_{t+2}$. But we want the present value of that income in period *t*. To get the present value at time *t* of resources we have at time $t+1$, we discount by $1+R_{t+1}$. So the present value at time *t* of y_{t+2} is given by $\frac{1}{(1+R_{t+1})(1+R_{t+2})} y_{t+2}$. In a three-period model, the lifetime budget constraint is given by:

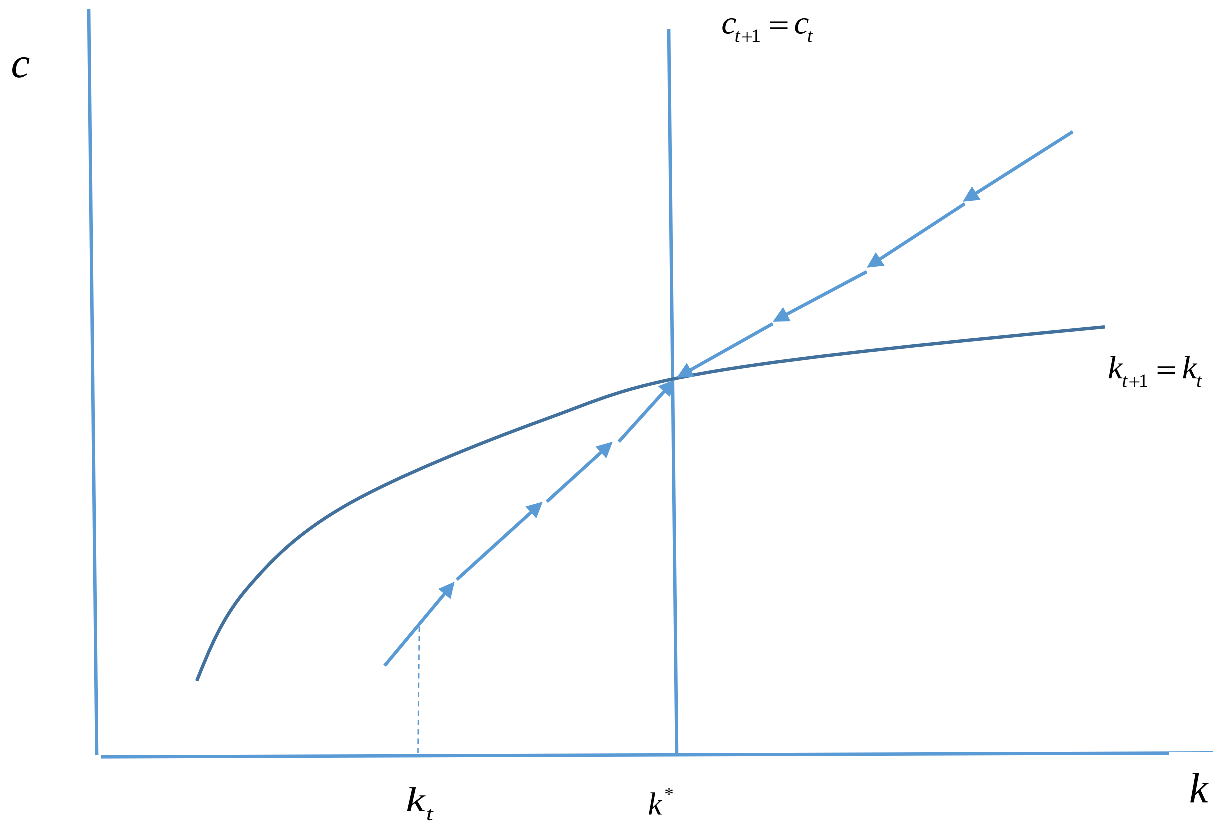
$$c_t + \frac{1}{1+R_{t+1}} c_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} c_{t+2} = y_t + \frac{1}{1+R_{t+1}} y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} y_{t+2} .$$

In our model, time goes on forever. We write the lifetime budget constraint as:

(12) $$c_t + \frac{1}{1+R_{t+1}} c_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} c_{t+2} + \ldots = y_t + \frac{1}{1+R_{t+1}} y_{t+1} + \frac{1}{(1+R_{t+1})(1+R_{t+2})} y_{t+2} + \ldots$$

where the triple dots represent the infinite discounted sums.

One can show mathematically (although not here!) that the only path for consumption and the capital stock that satisfies the capital accumulation equation, (5), the Euler equation, (7), and the lifetime budget constraint, (12), is the saddle path. The economy must be on the saddle path, where consumption and capital head toward the steady state.



The graph above shows how consumption and capital evolve if the initial capital stock is given by k_t. The economy takes k_t as given when period *t* starts, because the capital stock at any time is determined by saving and capital accumulation in previous periods. Given k_t, the consumption level must be on the saddle path in order for the economy to be following its optimal saving and capital accumulation plan, and satisfying its lifetime budget constraint. Over time, consumption and capital rise toward the steady state.

Along the path toward the steady state, the economy is saving and accumulating capital. But its consumption is rising over time as income rises.

If the economy had started with a capital stock greater than the steady state, then it would be on the saddle path to the right of the steady state. Consumption and capital would fall toward the steady state.

The lessons we have learned so far from this model:

- We still have a steady state with no long run growth!
- Saving more this period increases the capital stock next period.
- As the capital stock increases, the marginal product of capital falls.
- Output approaches a steady state.

The optimal consumption path when below the steady state is to have high consumption growth initially. But as the marginal product of capital falls, consumption growth falls, and consumption growth approaches zero as R goes toward $\frac{1-\beta}{\beta}$.

That is, if the economy starts with capital below the steady state, it has a high marginal product of capital, and high values of R_t, relative to the steady state. Those high returns give households the incentive to save and accumulate capital. As capital accumulates, however, the marginal product of capital falls, and the incentive to save falls. In the long run, the incentive to save dies out as R goes toward $\frac{1-\beta}{\beta}$.

As in the basic Solow model, it is the diminishing marginal product of capital that is responsible for growth dying out. In that model, growth died out because as the marginal product of capital falls, the marginal additions to capital through saving a constant share of output fall until new investment just equals the amount of old capital that is depreciating. In this model, the diminishing marginal product of capital reduces the return to saving, and thereby diminishes the incentive to save, until finally according to the Euler equation, households are satisfied with their level of consumption and $R = \frac{1-\beta}{\beta}$.

Changes in *A* and β

How does the economy evolve when there is a permanent increase in TFP? We have seen already that steady state capital per worker and steady state consumption increase. How does the economy approach the new steady state?

We can answer that by asking how a change in *A* affects the phase diagram.

First, the equation for $c_{t+1} = c_t$ is given by $k_t = (\beta\alpha A)^{\frac{1}{1-\alpha}}$. (See the derivation of equation (9).) An increase in *A* clearly will shift the vertical line where $c_{t+1} = c_t$ to the right.

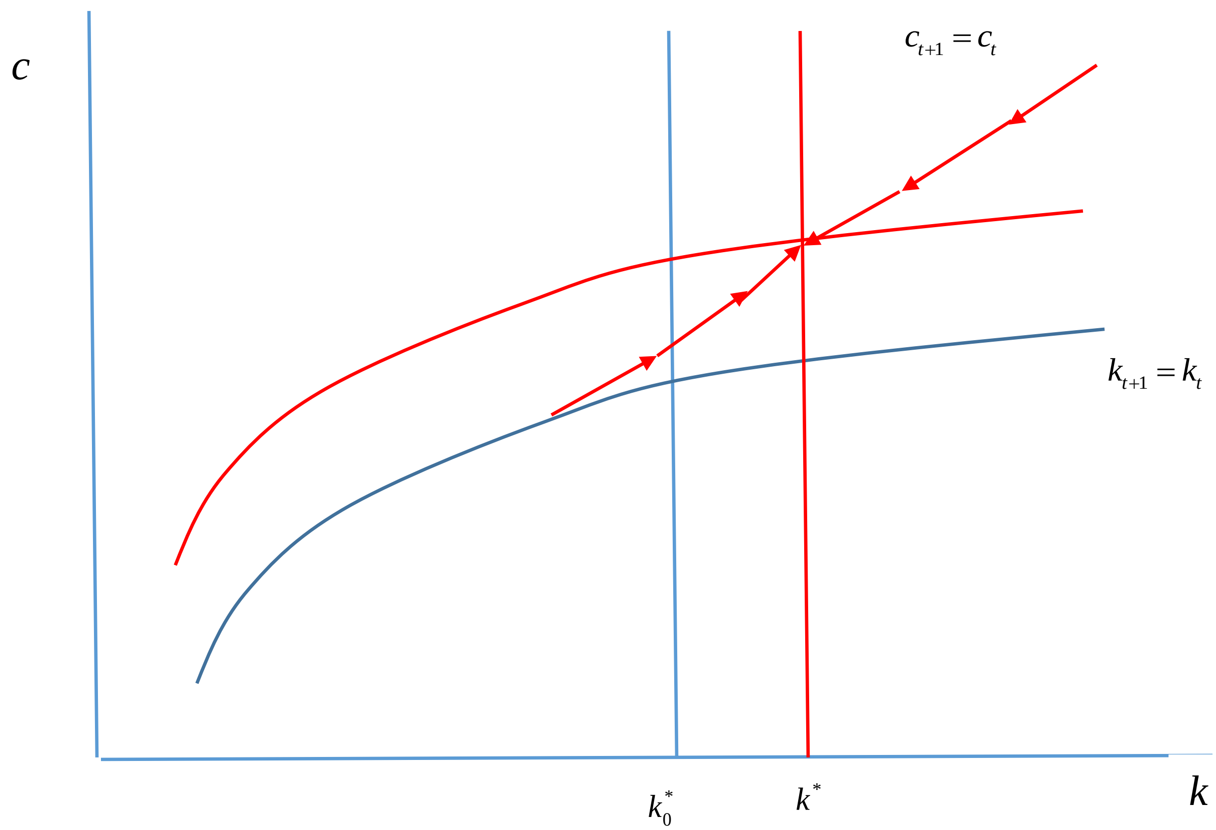
The equation for the curve where $k_{t+1} = k_t$ is given by $c_t = Ak_t^{\alpha}$. An increase in *A* shifts that curve up.

The graph below shows the shifts in those two curves. The blue lines are the original curves, and the red lines are the new curves.

Also included is the *new* saddle path (the old saddle path is omitted to reduce clutter in the graph.)

Suppose the economy was starting at the old steady state when TFP increased. We see that consumption initially increases. But also saving increases because the increase in TFP has raised the return to capital. Along the transition toward the new steady state, the economy saves, but also enjoys increasing consumption.

Note that the path is drawn in such a way that consumption initially increases. That is not necessarily the case. It may be that the increase in the return to capital is sufficiently high that consumption falls initially when TFP increases. That is, we know $\frac{c_{t+1}}{c_t}$ becomes positive, but this might be accomplished not only by an increase in c_{t+1}, but also by a drop in c_t.


c
$c_{t+1} = c_t$
$k_{t+1} = k_t$
k_0^*
k^*
k

Next we consider a change in the patience of consumers – an increase in the utility discount factor, β.

We have seen that an increase in β increases the steady state capital per worker, output per worker and consumption. How is that accomplished? That is, what is the dynamic path toward the new steady state?

First, as we have seen, the equation for $c_{t+1} = c_t$ is given by $k_t = \left(\frac{\beta}{1-\beta}\alpha A\right)^{\frac{1}{1-\alpha}}$. An increase in β will shift the vertical line where $c_{t+1} = c_t$ to the right.

The equation for the curve where $k_{t+1} = k_t$ is given by $c_t = Ak_t^{\alpha}$. An increase in β has no effect on this line.

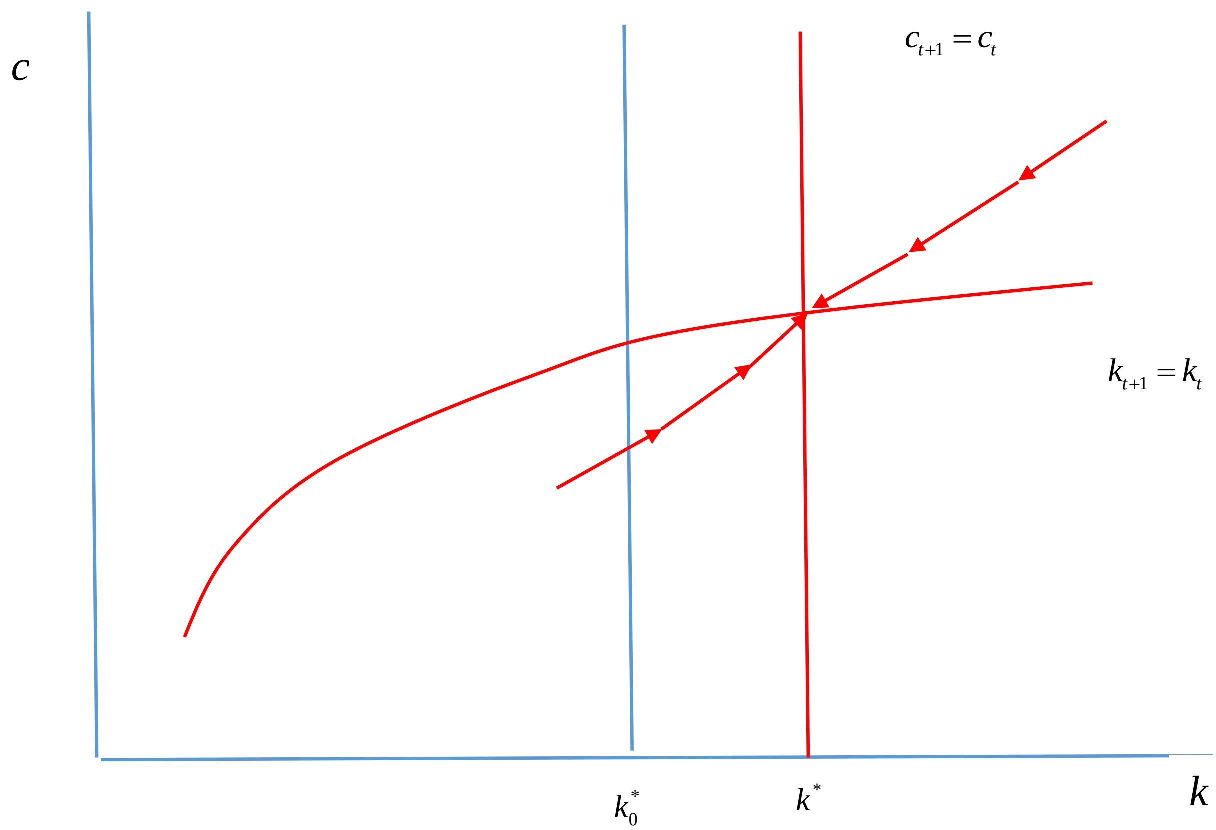
The graph below shows how the increase in patience affects the phase diagram. The red lines represent the new curves, though of course the curve for $k_{t+1} = k_t$ is the same as the old one.

Suppose the economy is starting at the previous steady state, where the $k_{t+1} = k_t$ curve intersects the blue vertical line. It is clear from the phase diagram that the increase in patience leads the economy initially to consume less. Consumption must drop initially so the economy is on the saddle path toward the new steady state.

Eventually consumption rises above its initial level as the economy accumulates capital and income increases.

In the basic Solow model, there was a question of whether the initial drop in consumption caused by an increase in the saving rate was optimal. In that model, consumption falls initially, and then rises in the long run. We can question whether such a trade-off improves utility or not – a short-run loss in consumption in exchange for a longer-run gain. There is one case where the tradeoff is clearly not worth it, which is when consumption is below the golden rule level of consumption. In that case, increasing the saving rate turns out to lower consumption both in the short run and in the long run.

In this model, there is no question about whether the tradeoff is a good one. That is because we have solved for the optimal path of consumption. The fact that it is "optimal" means it is the best path possible – the path that maximizes utility subject to the lifetime budget constraint, and subject to the equation that determines the accumulation of capital.



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k_0^*
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