Econ 730
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International Financial Economics

Charles Engel

Lecture 11: Portfolio Choice
We examine the portfolio choice of an investor who wants a high expected return on his portfolio, but dislikes variance.

We then incorporate that model of asset choice into a general equilibrium model that determines the return on assets.

We will first look at the general equilibrium of the closed economy, and derive the famous results from CAPM (Capital Asset Pricing Model.)

Then we will go on to the harder problem of equilibrium in an international model.

One of the things we will ultimately get out of this chapter is a model for the foreign exchange risk premium.
Use the logarithmic approximation to define the risk premium as:

\[ \rho_t = i_t^* - i_t + E_t(s_{t+1} - s_t). \]

This is the risk premium or expected “excess return” on the foreign asset expressed in terms of nominal interest rates (foreign minus home) and the expected change in the log of the nominal exchange rate.

We will assume that home and foreign consumer prices are constant. This simplification means real rates of return and nominal rates of return are the same. We will express all returns in real terms, so the above expression will be written as:

\[ \rho_t = r_t^* - r_t + E_t(q_{t+1} - q_t). \]
Portfolio Choice

Consider an investor (investor $i$) that wants to choose a portfolio this period to maximize:

$$E(W_{i,+1}) - \alpha_i \text{ var}(W_{i,+1})$$

$W_{+1}$ is the investor’s wealth next period, which depends on the returns on his portfolio. He likes higher expected wealth next period, $E(W_{i,+1})$, but dislikes variance, $\text{var}(W_{i,+1})$.

The parameter $\alpha_i$ measures his dislike of variance compared to expected return. It is a measure of his “risk aversion”. More uncertainty increases the variance of next period’s wealth.
The investor’s wealth next period will be given by:

\[(1) \quad W_{i,+1} = rX_{i,0} + r_1X_{i,1} + r_2X_{i,2} + \ldots + r_nX_{i,n} + W_i,\]

where \(X_i\) is the amount invested this period in each of the \(n+1\) assets. The returns, \(r_i\), are not known today when we make our investment.

(Asset 0 will take on particular importance below. For now, it is just another asset, whose return is \(r\), but later we will assume \(r\) is the “riskless” asset whose return is known now.)

\(W_i\) is the investor’s initial wealth – his wealth today.
If we let $W$ equal today’s wealth, the budget constraint for choosing the investment amounts is

$$W_i = X_{i,0} + X_{i,1} + X_{i,2} + \ldots + X_{i,n}.$$ 

We can rewrite this constraint, dividing both sides of the equation by $W$:

(2) \[ 1 = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2} + \ldots + \lambda_{i,n}. \]

Here, $\lambda_{i,j} = \frac{X_{i,j}}{W_i}$ is the share of initial wealth invested in asset $j$. Then we have:

$$W_{i+1} = W_i \left( r \lambda_{i,0} + r_1 \lambda_{i,1} + r_2 \lambda_{i,2} + \ldots + r_n \lambda_{i,n} \right) + W_i.$$
Then we can see:

\[ EW_{i,+1} = W_i E \left( r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n} \right) + W_i, \]  

and

\[ \text{var}(W_{i,+1}) = W_i^2 \text{var}( r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n} ). \]

The investor’s goal is to maximize, therefore

\[ W_i E \left( r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n} \right) \]

\[ -\alpha_i W_i^2 \text{var}( r\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n} ) + W_i \]  

(3)
Maximizing this is the same as maximizing

\[ (4) \]

\[ E(\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n}) - \alpha_i \text{ var}(\lambda_{i,0} + r_1\lambda_{i,1} + r_2\lambda_{i,2} + \ldots + r_n\lambda_{i,n}) \],

where \( a_i \equiv \alpha_i W_i \), because \( W \) is known today, so dividing equation (3) by \( W_i \) does not change the optimal choices.

The investor then needs to choose the asset shares \( \lambda_{i,j} \) to maximize equation (4) subject to the constraint (2).
Now let asset 0 be an asset that has no risk so the return known with certainty at the time of investment.

So no need for an expectation sign in front of the return. Also, $r$ does not contribute to the variance of the return on the portfolio. We can simplify (4) as:

\[(5) \quad r \lambda_{i,0} + E \left( r_1 \lambda_{i,1} + r_2 \lambda_{i,2} + \ldots + r_n \lambda_{i,n} \right) - \alpha_i \text{ var} \left( r_1 \lambda_{i,1} + r_2 \lambda_{i,2} + \ldots + r_n \lambda_{i,n} \right)\]

Let's now define $\lambda_i$ to be the sum of all of the investment shares on the risky assets:
(6) \[ \lambda_i = \sum_{j=1}^{n} \lambda_{i,j}. \]

Because the total shares add up to one, then we must have \( \lambda_{i,0} = 1 - \lambda_i \).

Now define \( \mu_{i,j} \) to be the share of the portfolio of risky assets only (the portfolio of assets that are comprised of assets 1 through \( n \), whose returns are risky.) We have:

(7) \[ \mu_{i,j} \equiv \frac{\lambda_{i,j}}{\lambda_i}. \]

Let \( r_{i,m} \) be the return on the investor’s portfolio of risky assets:
(8) \( r_{i,m} = r_{1}\mu_{i,1} + r_{2}\mu_{i,2} + \ldots + r_{n}\mu_{i,n} \).

Then we have

(9) \( E\left( r_{i}\lambda_{i,1} + r_{2}\lambda_{i,2} + \ldots + r_{n}\lambda_{i,n} \right) = \lambda_{i}E\left( r_{1}\mu_{i,1} + r_{2}\mu_{i,2} + \ldots + r_{n}\mu_{i,n} \right) = \lambda_{i}Er_{im} \).

Also, we have

\[
\text{var}\left( r_{i}\lambda_{i,1} + r_{2}\lambda_{i,2} + \ldots + r_{n}\lambda_{i,n} \right) = \lambda_{i}^{2}\text{var}\left( r_{1}\mu_{i,1} + r_{2}\mu_{i,2} + \ldots + r_{n}\mu_{i,n} \right) = \lambda_{i}^{2}\text{var}(r_{i,m})
\]

Substitute equations (9) and (10) into the objective, (5), and also substitute the constraint \( \lambda_{i,0} = 1 - \lambda_{i} \), so that we can rewrite equation (5) as:
We are breaking the optimization problem of the investor into two parts. First, he chooses \( \lambda_i \), the share of his overall portfolio that will be invested in risky assets. The remainder, \( 1 - \lambda_i \) will be invested in the risk-free deposit that pays a return \( r \).

Then he chooses the weights \( \mu_{i,j} \) that each of the risky assets will get in his portfolio of risky assets.

The first-order condition for choosing \( \lambda_i \) to maximize (11) is:

\[
-r + E(r_{i,m}) - 2a_i \lambda_i \text{ var}(r_{i,m}) = 0,
\]

\[\text{(11)} \quad r(1 - \lambda_i) + \lambda_i E(r_{i,m}) - a_i \lambda_i^2 \text{ var}(r_{i,m}).\]
which can be solved to give us:

\[
\lambda_i = \frac{E(r_{i,m}) - r}{2a_i \text{var}(r_{i,m})}.
\]

Holding the denominator constant, the investor puts a greater share into the risky portfolio when the return on the risky portfolio rises relative to the riskless return (when \(E(r_{i,m}) - r\)).

He invests less in the risky portfolio, holding other things constant, when the variance of the risky portfolio, \(\text{var}(r_{i,m})\), is higher.

An investor with the higher degree of risk aversion (higher \(a_i\)) will invest a smaller share in the risky portfolio.
Next, the problem of choosing the optimal risky portfolio. The investor still wants to choose his portfolio to maximize the expression in equation (11). We can substitute our solution for the optimal value of $\lambda_i$ into equation (11), and we find equation (11) can be rewritten as:

\[(13) \quad r + \frac{(E(r_{i,m}) - r)^2}{4a_i \text{ var}(r_{i,m})}. \]

The investor will now choose the weights of his portfolio of risky assets, $\mu_{i,j}$, to maximize the expression in equation (13). However, the choice of variables to maximize any function also maximizes any linear function of the original function.
The choice of $\mu_{i,j}$ that maximizes (13) also maximizes:

\[ \frac{(E(r_{i,m}) - r)^2}{\text{var}(r_{i,m})} \tag{14} \]

This expression is simply a linear function of the function in (13) (in which we subtract $r$ from the expression in (13) and multiply by $4a_i$.)

We have derived one of the most important results in investment theory. The investor wants to choose a risky portfolio – choose the values of the weights, $\mu_{i,j}$ - to maximize the expression in (14).

The choice of the $\mu_{i,j}$ that maximizes (14) does not depend on $a_i$! All investors choose the same $\mu_{i,j}$. 
Suppose the world was full of risk-averse investor who only differed in their degree of risk aversion, \( a_i \). All of these investors will choose exactly the same weights in their portfolio of risky assets. Why?

Because they are all choosing their \( \mu_{i,j} \) to maximize (14), but (14) does not depend on their degree of risk aversion. All investors choose \( \mu_{i,j} \) to maximize exactly the same function.

The only way in which the degree of risk aversion will affect the portfolio choice is through its effect on \( \lambda_i \), the share of wealth invested in risky assets.
As equation (12) shows, more risk averse investors put a smaller share of their wealth in the risky portfolio, and a greater share in the riskless asset.

Now consider the allocation of risky assets to maximize expression (13), keeping in mind that $r_{i,m}$ is given by equation (8). With a little bit of work, we can derive from the first-order condition for choosing $\mu_{i,j}$ the following relationship:

\[
E(r_j) - r = \frac{E(r_m) - r}{\text{var}(r_m)} \text{cov}(r_j, r_m).
\]

(15) 

Notice that we dropped the subscript $i$ from the return on the investor’s risky portfolio – we’ve written $r_m$ instead of $r_{i,m}$.
We will use equation (15) below, so let’s set it aside where it will be handy. Also, we can substitute in for $\frac{E(r_m - r)}{\text{var}(r_m)}$ from equation (12) to write:

\[(16) \quad E(r_j) - r = 2a_i \lambda_i \text{cov}(r_j, r_m).\]

Since $\frac{E(r_m) - r}{\text{var}(r_m)}$ does not depend on the investor $i$, it must be that $a_i \lambda_i$ is the same for all investors, a fact which we should hold onto until later.

Equation (16) comes from the first-order condition for choosing asset $j$. Since there are $n$ risky assets, there will be $n$ such first-order conditions.
Each is a function of all of the $\mu_{i,j}$ because:

$$\text{cov}(r_j, r_m) = \text{cov}(r_j, r_1\mu_{i,1} + r_2\mu_{i,2} + \ldots + r_n\mu_{i,n})$$

$$= \mu_{i,1} \text{cov}(r_j, r_1) + \mu_{i,2} \text{cov}(r_j, r_2) + \ldots + \mu_{i,n} \text{cov}(r_j, r_n)$$

In vector notation, we could write

$$E(\bar{r}) - r = 2\alpha_i \lambda_i \Omega \mu_i,$$

where

$$\bar{r} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}, \quad \Omega = \begin{bmatrix} \text{var}(r_1) & \text{cov}(r_1, r_2) & \ldots & \text{cov}(r_1, r_n) \\ \text{cov}(r_2, r_1) & \text{var}(r_2) & \ldots & \text{cov}(r_2, r_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(r_n, r_1) & \text{cov}(r_n, r_2) & \ldots & \text{var}(r_n) \end{bmatrix}, \quad \mu_i = \begin{bmatrix} \mu_{i,1} \\ \mu_{i,2} \\ \vdots \\ \mu_{i,n} \end{bmatrix}$$
(17) \[ \mu_i = \frac{1}{2 a_i \lambda_i} \Omega^{-1} \left[ E(\bar{r}) - r \right] \]

Since \( a_i \lambda_i \) is the same for all investors, they all hold the same risky portfolio. Substituting for \( \lambda_i \) from equation (12), we find:

(18) \[ \mu_i = \frac{\text{var}(r_m)}{E(r_m) - r} \Omega^{-1} \left[ E(\bar{r}) - r \right]. \]

Because all investors choose the same risky portfolio, in the next section, we describe \( r_m \) as the return on the “market portfolio”. There we see that \( \frac{\text{var}(r_m)}{(E(r_m) - r)} = \frac{1}{2\bar{a}} \), where \( \bar{a} \) is a measure of the average risk aversion in the economy.
Equilibrium in a Closed Economy

We investigate how asset returns are determined in an economy that is populated by investors that are like the ones in the previous section. The investors in the economy differ only in their degree of risk aversion, as captured in the parameter $a_i$.

We have already seen that the only way in which the investors’ portfolios differ is in the division of their portfolio between the riskless asset and the risky portfolio. Investors’ choice of $\lambda_i$, the share of their wealth invested in the portfolio of risky assets, depends on the degree of risk aversion. The larger is $a_i$, the smaller is $\lambda_i$. In fact, we noted in the last section that $a_i \lambda_i$ is the same for all investors.
We also noted that the allocation of each asset as a share of the risky portfolio is the same for all investors. The allocation does not depend on their degree of risk aversion. The $\mu_j$ from last section are the same for all investors.

Equation (15), derived from each investor’s first-order conditions, is the same for all investors because they all choose the same risky portfolio. We repeat that equation here for convenience:

$$E(r_j) - r = \frac{E(r_m) - r}{\text{var}(r_m)} \text{cov}(r_j, r_m).$$

This equation can be considered an equilibrium relationship that determines the expected return on asset $j$. 

\[ \text{(19)} \]
We will refer to $r_m$ as the return on the “market portfolio.” Since all investors hold the same risky portfolio, the weights of each asset in the risky portfolio must be equal to the value of that asset as a share of the value of all risky assets.

For example, if one percent of every investor’s risky portfolio is invested in Google stock, then one percent of risky investments for the economy as a whole must be in Google stock. So, Google’s share in the risky portfolio is equal to its value as a share of the value of all risky assets. We call it the market portfolio because the weight given to each asset is determined by the market value of that asset as a share of the total market value of risky assets.
A very common and almost famous way of writing equation (19) is:

\[
E(r_j) - r = \beta_j (E(r_m) - r), \text{ where } \beta_j \equiv \frac{\text{cov}(r_j, r_m)}{\text{var}(r_m)}.
\]

Among financial economists, \(\beta_j\) is called asset \(j\)’s “beta”. That term is widely used on Wall Street.

The formula for \(\beta_j\) is the formula for the slope coefficient in a regression of \(r_j\) on \(r_m\). Usually analysts measure \(r_m\) as the return on some broad index of equities, such as the S&P 500.
What is the intuition of this result? Equation (20) says that asset \( j \) has a higher expected return the larger is its \( \beta_j \). Different assets have different betas depending on the covariance of the return on the asset with \( r_m \). Assets that have a higher covariance with \( r_m \) have a higher expected return, which means the market perceives that asset as riskier.

The market insists on a higher expected rate of return in order to compensate investors for the risk they perceive in holding the asset. We can conclude that the market believes that the appropriate measure of risk is \( \text{cov}(r_j, r_m) \).

To see why the market takes \( \text{cov}(r_j, r_m) \) as the measure of the risk of asset \( j \), write out the variance of the return on the market portfolio, \( r_m \).
If the investor increases its holdings of, for example, asset 1, we can calculate the effect on the variance:

\[
\frac{d \text{var}(r_m)}{d \mu_1} = 2 \left( \mu_1 \text{var}(r_1) + \mu_2 \text{cov}(r_1, r_2) + \ldots + \mu_n \text{cov}(r_1, r_n) \right) \\
= 2 \text{cov}(r_1, \mu_1 r_1 + \mu_2 r_2 + \ldots + \mu_n r_n) \\
= 2 \text{cov}(r_1, r_m)
\]
We can see from this calculation that \( \text{cov}(r_1, r_m) \) measures the influence of adding a little bit more of asset 1 to the variance of the return on the market portfolio.

An asset is considered risky by the market – and so deserves a higher expected return – if adding a little of that asset contributes to the overall variance of return on the portfolio. Since investors are averse to higher variance on the return to their portfolio, this is the appropriate measure of the riskiness of an asset.

Most assets have returns that tend to be positively correlated with \( r_m \). When the overall economy falls into recession, the returns on stocks, housing, bonds and most other assets tends to decline, but they tend to rise together during booms. An asset is considered riskier the more it increases the variance of the portfolio.
The covariance of the return of an asset with the return on the portfolio is determined both by the correlation of the return, and the volatility of the return. That is, \( \text{cov}(r_j, r_m) = \text{corr}(r_j, r_m) \sqrt{\text{var}(r_j) \text{var}(r_m)} \), where \( \text{corr}(r_j, r_m) \) refers to the correlation. An asset is riskier if, holding the variance of the return constant, its return has a higher correlation with the market return. Also the asset is riskier if holding the correlation constant, the variance of its return increases.

While equation (20) is the most common way of writing expression (19), it is not an entirely satisfactory way of explaining the risk premium on asset \( j \), because it leaves open the question of what determines the expected excess return on the market portfolio, \( E(r_m) - r \). That issue can be resolved by referring back to equation (16), repeated here for convenience:
(21) \[ E(r_j) - r = 2a_i \lambda_i \text{cov}(r_j, r_m). \]

Recall that we derived this equation by using equation (12), which determined investor \( i \)'s demand for risky assets, where \( \lambda_i \) is the share of the overall portfolio that will be invested in the market portfolio. We have noted that since \( E(r_j) - r \) and \( \text{cov}(r_j, r_m) \) are the same for all investors, then \( a_i \lambda_i \) must be the same for all investors.

In a closed economy, the total holdings of the riskless deposit must add up to zero. For every dollar that is lent there must be a dollar that is borrowed. If we add up every investors’ holdings of the riskless asset, they must add to zero.
(1 – λᵢ)Wᵢ is investor i’s holding of the riskless asset – it is the share of his portfolio invested in the riskless asset times his initial wealth, which gives the amount he invested in the riskless asset. Add that up across all investors, and we must get zero: \[ \sum (1 – \lambda_i)W_i = 0. \]

We can rewrite this as \[ \sum W_i = \sum \lambda_i W_i. \]

Let \( W \) be the total wealth in the economy, and let \( s_i \) be investor i’s share of total wealth. We can then write

\[ 1 = \sum_i \lambda_i s_i. \]  

Now rearrange equation (21), so that we have the solution for the share of risky assets on one side of the equation:
\[ \lambda_i = \frac{E(r_j) - r}{2 \text{cov}(r_j, r_m) a_i} \cdot \]

Multiply both sides by \( s_i \) and sum up over all of the agents:

\[ (23) \quad 1 = \sum_i s_i \lambda_i = \frac{E(r_j) - r}{2 \text{cov}(r_j, r_m)} \sum_i s_i a_i. \]

From this equation, we can now rearrange things one more time to get:

\[ (24) \quad E(r_j) - r = 2\bar{a} \text{cov}(r_j, r_m). \]

In this equation, \( \bar{a} \) is defined as:
\[
\bar{a} = \frac{1}{\sum_i s_i}.
\]

It is a measure of the average risk aversion in the economy. This may seem like an odd way to measure average risk aversion – it is the inverse of a weighted average of the inverse of investors’ risk aversion coefficients, with the weights being given by each investor’s share of national wealth.

If everybody had the same degree of risk aversion, call it \( a \), then our measure of average risk aversion would just equal \( a \). Or, if everybody’s wealth share were the same, then our measure would be 1 divided by the average value of \( 1/a_i \).
We can see that in equilibrium, equation (24) tells us that the risk premium on asset $i$ (the expected return in excess of the riskless rate) is determined by only two things: the average level of risk aversion, $\bar{a}$, and the contribution of asset $i$ to the volatility of the market portfolio, given by $\text{cov}(r_i, r_m)$. We’ve done a lot of algebra to arrive at a conclusion that seems so intuitive and almost obvious!

Let $r_i$ in equation (24) refer to the return on the entire portfolio, $r_m$, and we get:

\[
(26) \quad E(r_m) - r = 2\bar{a} \text{ var}(r_m).
\]

The risk premium on the market portfolio is determined by the level of risk aversion, and the variance of returns on the market portfolio.
Open Economy

Finding the market equilibrium in an open economy is much harder than in a closed economy. We can no longer assume that the only way that investors around the world differ is in their degree of risk aversion. There is another crucial difference – home investors value returns in terms of home units of consumption (or, when there is no inflation uncertainty, they value returns in home currency units), while foreign investors care about returns in units of foreign consumption (or foreign currency when there is no inflation uncertainty.)

Our goal is to understand the foreign exchange risk premium.
Let’s simplify the problem in the following ways. First, assume that all investors in the home country and foreign country are equally risk averse. In particular, we will assume that $a_i$ is the same for all investors in both countries, so that we will not distinguish between investors based on their relative preference for save versus risky assets.

However, we do allow for a very important difference between home and foreign investors. Home investors pay prices set in the home currency, and so are concerned about the mean and variance of returns expressed in home currency units, while foreign investors are concerned about returns in foreign currency units.

We also will simplify the model by assuming that investors may choose between three assets – a home bond, a foreign bond, and a stock.
The home bond pays a return $r$ in home currency units, and the foreign bond pays a return $r^*$ in foreign currency units. These returns are known at the time the investor makes his portfolio choice. The home bond, therefore, is not risky for the home investor and the foreign bond is not risky for the foreign investor. But, the foreign bond is risky for the home investor because of exchange-rate risk, and vice-versa.

What determines the foreign exchange risk premium? If we think in equilibrium the riskier asset should pay the higher rate of return, which should pay the higher rate of return?

Maybe neither – maybe the risk premium is zero and UIP holds. Maybe the concerns of the Home and Foreign investors balance out. We will consider here a simple static model. If net debt is zero, maybe nobody takes a position (long or short) in either deposit.
In our simple open-economy model, as in CAPM, we do not consider full general equilibrium. We take the probability distribution of returns as given. We also take initial wealth as given. And, as mentioned already, we assume all goods prices are fixed in local currencies.

We know from standard CAPM that one of the determinants of excess return on asset $i$ is covariance of the return with the other assets in the investor’s portfolio. But the covariance of the return on a bond with the return on the equity is also something that needs to be treated carefully in the international setting.
For Home investors, the return on the foreign bond is, approximately, $i^* + s_{+1} - s$. The return on the equity is given by $r_x$, which is stochastic. The covariance of the returns of foreign bonds with equities for the home investor is given by $\text{cov}(s_{+1}, r_x)$. That is, the stochastic part of the return on the foreign bond is the exchange rate, and when it is higher, the foreign bond pays a higher return.

For the Foreign investor, the return on the home bond is approximately $i - (s_{+1} - s)$. The return on the equity is $r_x^* = r_x - (s_{+1} - s)$. The covariance of returns of the home bonds with equities for the Foreign investor is

\begin{equation}
\text{cov}(s_{+1}, r_x^*) = \text{cov}(s_{+1}, r_x - s_{+1}) = -\text{cov}(s_{+1}, r_x) + \text{var}(s_{+1}).
\end{equation}
Notice, then that the covariance of the exchange rate with the return on the equity has different implications for the Home and the Foreign investor. Suppose $\text{cov}(s_{+1}, r_x) > 0$. For the Home investor, when the exchange rate goes up (the home currency depreciates, and the foreign currency appreciates), the returns on the two risky assets tend to move together. Their returns are positively correlated. That is, there is “covariance” risk for Home investors holding foreign bonds. (Positive covariance is risky, because positive covariance adds to the riskiness of a portfolio if an investor is long in both assets.)

For the Foreign investor, things are not so clear in this case. If the foreign bond is risky for Home, does that mean necessarily that the home bond is a hedge for Foreign (i.e., $\text{cov}(-s_{+1}, r_x^*) < 0$)? No, because if $\text{var}(s_{+1})$ is large enough, we could still have $\text{cov}(-s_{+1}, r_x^*) > 0$. 
The return on the Home portfolio can be expressed as a weighted average of the returns on the individual assets:

\[ \lambda_x r_x + \lambda_F (i^* + s_{+1} - s) + (1 - \lambda_x - \lambda_F) i = \lambda_x (r_x - i) + \lambda_F (i^* - i + s_{+1} - s) + i \]

The variance of the returns is given by:

\[ \text{var} \left( \lambda_x (r_x - i) + \lambda_F (i^* - i + s_{+1} - s) + i \right) = \]
\[ \lambda_x^2 \text{var}(r_x) + \lambda_F^2 \text{var}(s_{+1}) + 2\lambda_x \lambda_F \text{cov}(r_x, s_{+1}) \]

Home investors want to maximize a function of the mean and variance of end-of-period wealth. They choose \( \lambda_x \) and \( \lambda_F \) to maximize, which is given by:
\[ E(\lambda_x (r_x - i) + \lambda_F (i^* - i + s_{+1} - s) + i) + \\
\frac{1}{2} a \var(\lambda_x (r_x - i) + \lambda_F (i^* - i + s_{+1} - s) + i) \]

From the first-order conditions, one can find:

(28) \[ \lambda_x = \frac{1}{a} \left[ \frac{\var(s_{+1}) E(r_x - i) - \cov(r_x, s_{+1}) E(i^* + s_{+1} - s - i)}{\var(r_x) \var(s_{+1}) - (\cov(r_x, s_{+1}))^2} \right] \]
This gives us an expression for the Home investor’s holdings of foreign bonds, and of the equity.

To derive the Foreign investor’s portfolio, we can simply use symmetry. That is, everywhere in the expression above where we have the return on the equity for the Home investor, we replace it with the return on the equity for the Foreign investor, and everywhere where we have the return on the foreign bond for the Home investor, we replace it with the return on the home bond for the Foreign investor.
As we examine this, we see that the share of equities in the Foreign portfolio is the same as the share of equities in the Home portfolio! That is a very important finding. Foreign and Home investors differ in the foreign exchange risk that they face. But foreign exchange risk only affects their demand for the bonds, not for the equity. That is, hedging of foreign exchange risk occurs through the portfolio of nominal bonds, where the risk arises only from foreign exchange fluctuations.

The Foreign investor’s holdings of the home bond are given by:
(30)

\[
\lambda^*_H = \frac{1}{a} \left[ \frac{\text{var}(r_x - s_{+1}) E(i - s_{+1} + s - i^*) - \text{cov}(r_x - s_{+1}, -s_{+1}) E(r_x - s_{+1} + s - i^*)}{\text{var}(r_x) \text{var}(s_{+1}) - (\text{cov}(r_x, s_{+1}))^2} \right]
\]

\[
= \frac{1}{a} \left[ \left[ \text{cov}(r_x, s_{+1}) - \text{var}(r_x) \right] E(i^* + s_{+1} - s - i) + \left[ \text{cov}(r_x, s_{+1}) - \text{var}(s_{+1}) \right] E(r_x - i) \right]
\]

\[
\frac{1}{a} \left[ \frac{\text{var}(r_x) \text{var}(s_{+1}) - (\text{cov}(r_x, s_{+1}))^2}{\text{var}(r_x) \text{var}(s_{+1}) - (\text{cov}(r_x, s_{+1}))^2} \right]
\]
Now, let’s use these portfolio demands to derive equilibrium rates of return. In this static model, neither Home nor Foreign can take on net debt. This implies $\lambda_x = \lambda_x^* = 1$.

For example, if the Home country has made loans to the Foreign household by buying the Foreign bond, so $\lambda_F > 0$, then it must have offset that by selling Home bonds to the Foreign household. If it were a net borrower, when would it ever repay its loans? So we must have that the share of the Home portfolio in Home bonds, $1 - \lambda_x - \lambda_F$ is minus the share in Foreign bonds, that is $1 - \lambda_x - \lambda_F = -\lambda_F$. But this implies $\lambda_x = 1$. Similar logic applies to the Foreign investor, hence we have $\lambda_x = \lambda_x^* = 1$.

Then imposing that $\lambda_x = 1$ in equation (28) gives us:
Next, let $\omega$ be Home’s share of world initial wealth: \[ \omega = \frac{W}{W + SW^*}. \]

We must have $\omega \lambda_F = -(1 - \omega) \lambda_F^*$, where we let $\lambda_F^*$ denote the Foreign country’s holdings of foreign bonds. This means that the Home lending in foreign currency bonds equals the Foreign borrowing in foreign currency bonds. That is, we are simply saying $W \lambda_F = -SW^* \lambda_F^*$.

As we have noted, because a country can be neither a net lender nor borrower, we have $\lambda_F^* = -\lambda_H^*$. Substituting this relationship into the equation that says $\omega \lambda_F = -(1 - \omega) \lambda_F^*$, we find $\omega \lambda_F = (1 - \omega) \lambda_H^*$. 

\[ (31) \quad 1 = \frac{1}{a} \left[ \frac{\text{var}(s_{+1}) E(r_x - i) - \text{cov}(r_x, s_{+1}) E(i^* + s_{+1} - s - i)}{\text{var}(r_x) \text{var}(s)_{+1} - (\text{cov}(r_x, s_{+1}))^2} \right]. \]
Since the denominators of the expressions for \( \lambda_F \) and \( \lambda^*_H \) are equal, we can use the numerators of equations (29) and (30), and the fact that \( \omega \lambda_F = (1 - \omega) \lambda^*_H \) to derive:

\[
\omega \{ \text{var} (r_x) E(i^* + s_{+1} - s - i) - \text{cov} (r_x, s_{+1}) E(r_x - i) \} = \\
(1 - \omega) \left\{ \left[ \text{cov} (r_x, s_{+1}) - \text{var}(r_x) \right] E(i^* + s_{+1} - s - i) \right\} \\
+ \left[ \text{cov} (r_x, s_{+1}) - \text{var}(s_{+1}) \right] E(r_x - i)
\]

(32)

Finally, we can use equations (31) and (32) to eliminate \( E(r_x - i) \) and find:

\[
E(i^* + s_{+1} - s - i) = a \left[ (\omega - 1) \text{var} (s_{+1}) + \text{cov} (r_x, s_{+1}) \right]
\]

(33)
For intuition, consider the thought experiment of letting Home have all the wealth, so \( \omega = 1 \). Then \( E(i^* + s_{+1} - s - i) = a \text{cov}(r_x, s_{+1}) \). The foreign bond pays a risk premium, but only because its return covaries positively with the equity, which is risky. That is, if Home has all the wealth, then it alone determines the equilibrium returns. It would consider the foreign bond risky, but only if the return on the foreign bond was positively correlated with the return on the equity. Indeed, if it had a negative correlation, then the expected return on the foreign bond would be less than \( i \): it would be as if the foreign bond were providing insurance for the fluctuations in the return on the equity.

If the foreign agent has all the wealth, then:

\[
E(i^* + s_{+1} - s - i) = a \left[ -\text{var}(s_{+1}) + \text{cov}(r_x, s_{+1}) \right] = a \text{cov}(s_{+1}, r_x^*). 
\]
In general, we find (remember, $r^*_x = r_x - (s_{+1} - s)$)

$$E(i^* + s_{+1} - s - i) = \omega a \text{cov}(s_{+1}, r_x) + (1 - \omega) a \text{cov}(s_{+1}, r^*_x)$$

In the end, then, the expected excess return depends on the riskiness of home and foreign bonds, which depends on the covariance of their return for Home and Foreign investors with the return on equities.

We can also find the solution for expected returns on equities:

$$E(r_x - i) = a \left[ \text{var}(r_x) + (\omega - 1) \text{cov}(r_x, s_{+1}) \right].$$
Now, we can substitute all these expressions for excess returns back into the portfolio demand equations, to find:

\[ \lambda_x = 1 \quad \text{and} \quad \lambda_F = \omega - 1. \]

Also, \[ \lambda^*_x = 1 \quad \text{and} \quad \lambda^*_H = -\omega. \]

\( \lambda_x + \lambda_F \) is the share of risky assets for Home investors, then we can conclude that risky assets as a whole have a weight equal to \( \omega \) for Home investor. For Foreign investors, risky assets have a weight equal to \( 1 - \omega \).

In the portfolio of risky assets, for Home investors, Foreign bonds get a weight of \( \frac{-(1-\omega)}{\omega} \), while equities get a weight of \( \frac{1}{\omega} \).
Now let’s return and try to interpret the portfolio shares:

\[ \lambda_F = \omega - 1 \quad \text{and} \quad \lambda^*_H = -\omega. \]

Investors in each country borrow in the other’s currency. Why?

First take the completely symmetric case where \( \omega = 1/2 \). By symmetry \( \text{cov}(s_{+1}, r_x) = \text{cov}(s^*_+ r^*_x) \) where \( s^* = -s \)

But \( \text{cov}(s^*_+, r^*_x) = -\text{cov}(s^*_+, r^*_x) + \text{var}(s_{+1}) \), so
\[ \text{cov}(s_{+1}, r_x) = \text{cov}(s^*_+, r^*_x) = \frac{1}{2} \text{var}(s_{+1}). \]
Then for Home investors, foreign bonds are risky, and for Foreign investors, home bonds are risky. Under symmetry, the expected returns on the two bonds are equal, so Home would hold fewer foreign bonds, and Foreign would hold fewer home bonds. But the net holdings of both bonds is zero, so Home must hold negative amounts of foreign bonds, and Foreign must hold negative amounts of home bonds.

Can we interpret UIP as the case of risk neutrality? No, we cannot literally have \( a = 0 \) in this model. But we can look at the limit:

\[
\lim_{a \to 0} E\left( i^* + s_{+1} - s - i \right) = 0.
\]

Finally, what if there had been no equity to trade, and investors had only the option of investing in Home or Foreign bonds?
In equilibrium, we would find that $\lambda_F = 0$ and $\lambda^*_H = 0$ and $E(i^* + s_{+1} - s - i) = 0$. Why?

First, suppose $E(i^* + s_{+1} - s - i) = 0$. Then for Home, the Foreign bond is risky but offers no excess expected return. For Foreign, the Home bond is risky, but offers no excess expected return. So, they would each choose not to hold the bond of the other country.

Each is satisfied taking a zero position in the other country’s bonds as long as $E(i^* + s_{+1} - s - i) = 0$. In short, if there is foreign exchange risk, investors in both countries can avoid it simply by not buying the security of the other country!