How To Sell in a Sequential Auction Market*

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Abstract

A seller with one unit of a good faces \( N \geq 3 \) buyers and a single competitor. Buy-
ers who do not get the good from the seller will compete in a second-price auction for
one other identical unit. In this setting, the seller should not use a standard auction.
Instead, we characterize the optimal mechanism for the seller and show that it can be
implemented by a modified third-price or first-price auction with transfers between the
seller and the two highest bidders. The optimal mechanism features allocation to the
buyer with the second-highest valuation and a withholding rule that depends on the
second- and third-highest valuations. We show that this withholding rule raises signi-
ficantly more revenue than would a standard reserve price. We also consider the novel
implications of sequential cross-mechanism spillovers for competition between sellers.

1 Introduction

Much of the literature on competing mechanisms is focused on markets where sellers with
identical goods choose their mechanisms simultaneously and buyers then select among them.
However, collections of goods are commonly sold in a sequence of single-object auctions.
Auction houses such as Sotheby’s and Christie’s for art objects, Richie Brothers for used
construction and farm equipment, and Indiana Auto Public Auctions for used cars sell goods
sequentially in English auctions. Banks sell foreclosed homes sequentially using open outcry,
ascending price auctions held at local courthouses. These auction houses typically give

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sellers some control over reserve prices but not over the order in which the objects are sold. Sellers on auction platforms such as eBay sell their goods individually in ascending, second-price auctions and these auctions are sequenced by their unique arriving and closing times. Similarly, supply contracts between upstream and downstream firms are often negotiated sequentially due to unique expiration dates. In this paper, we take some first steps towards an equilibrium analysis of competition among sellers in a sequential auction setting.

We consider the auction design problem of a seller who competes against a subsequent seller. Our focus is on the allocation externality that arises in this setting and the impact it has on the optimal auction. The revenue that a seller can collect from the buyers is constrained by her need to incentivize buyers to participate in her auction. In a sequential setting, these incentive constraints depend on the buyer’s outside option, which is his payoff from bidding in a subsequent auction. That payoff is endogenous: it depends on the valuations of other buyers and on which (if any) of them receives the first good. For example, if a seller decides not to allocate the good to the bidder with the highest value or to any bidder, then losing buyers face stronger competition in subsequent auctions. The failure to allocate the good efficiently creates a negative payoff externality on buyers by lowering their outside option (and a positive externality for the subsequent seller). Our primary goal in this paper is to examine how a seller can exploit this externality to mitigate the effects of competition and extract more surplus from the buyers.

Our model is simple. There are two sellers, each with a single unit of an identical good, who sell their units sequentially to $N \geq 3$ buyers. These buyers have unit demands. Their values are private and independently drawn from a common distribution $F$ with density $f$. Any buyer who fails to obtain the good from the first seller participates in the auction of the second seller. The second auction is a second-price auction with no reserve price. Buyers have a dominant strategy to bid their value in that auction. Thus, any information that buyers obtain about their competitors’ types from the first auction has no effect on their bidding behavior in the subsequent auction. The second seller is non-strategic and does not respond to the first seller’s choice of auction or disclosure of outcomes. Given these assumptions, the first seller’s problem consists of designing an allocation and pricing rule to maximize revenues.

Our first result is a characterization of the optimal direct mechanism. We are interested

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1Smith and Thanassoulis [27] provide a motivating case study.
in this mechanism because it establishes how much revenue a seller can achieve and, more importantly, how she can achieve it. We show that the seller’s design problem reduces to a revenue-maximization problem that can be solved using standard methods. However, the solution is quite different from the optimal mechanism of a monopoly seller. In the latter case, a seller maximizes revenues by allocating the good to the buyer with the highest reported value as long as that report is high enough. By contrast, in our model, the seller optimally allocates her good only when the second-highest report is large compared to the third-highest. This rule clearly cannot be implemented by a reserve price. The second difference is that the seller allocates the object (if at all) to the second-highest bidder rather than the highest. That policy, because it ensures that the highest-value buyer always participates in the second auction, eliminates the incentive that a buyer may have to underreport his value in the hopes of increasing the probability that the good is allocated to someone else (recall that allocation is more likely when the third-highest reported value is low) and thus reducing future competition. Note however that misallocation does not occur. If the good is allocated, then the highest two types get the two goods.

The intuition behind the optimal allocation rule is as follows. Recall that when there is only one seller, the maximum surplus that she can create by allocating her good is \( x(1) \), the highest value among the buyers. (The seller’s value of the object is zero.) However, because the values of the buyers are private information, the most that the seller can extract is \( \psi(x(1)) \), where \( \psi(x) \) is the virtual value of a buyer with value \( x \).\(^2\) Thus, the optimal allocation rule in the single seller case is to allocate only if \( \psi(x(1)) \) is positive. When there is a second seller, the maximum surplus that the first seller can extract from the buyers is the difference between the surpluses generated from allocating the good and from not allocating it. If the first good is not allocated, then the second good will go to the buyer with the highest value at a price equal to the second-highest value, yielding a buyer surplus of \( x(1) - x(2) \). If the highest-value buyer gets the first object, then the second object will go to the buyer with the second-highest valuation at a price equal to the third highest, and total surplus for the buyers is \( x(1) + x(2) - x(3) \). (Note that the same total surplus results if the first object goes to the buyer with the second-highest value, since then the highest-value buyer gets the second object.) The difference is \( 2x(2) - x(3) \). As in the monopoly case, the seller cannot extract all of that surplus because of the private information of the buyers.

\(^2\)More precisely, \( \psi(x) \equiv x - (1 - F(x))/f(x) \).
Instead, we show that she can get $x(2) + \psi(x(2)) - x(3)$. The optimal rule, then, is to allocate whenever $x(2) + \psi(x(2)) - x(3)$ is positive and not otherwise.

Our second result is that the optimal mechanism can be implemented by a modified third-price auction or by a pay-your-bid auction in which the highest bidder gets a rebate equal to the price of the second good ($x(3)$ if the first good is allocated, $x(2)$ if it is not). In the third-price auction, buyers have a dominant strategy to report their type truthfully; in the pay-your-bid auction, the equilibrium is in monotone bid functions. Consequently, in each of these auctions, the seller can use the bids to implement the optimal allocation rule. The unusual feature of these auctions is that both the highest and second highest bidder make payments to the seller when the good is allocated. In the pay-your-bid auction, they simply pay their bids and, in the third-price auction, their payments are based on the third-highest bid. The intuition for why the seller can extract payments from the two highest bidders is that both benefit from the good being allocated and are willing to pay to ensure that this event occurs.

The expected revenue of the optimal auction in which the seller must sell the good with probability one is simply the expected value of the third order statistic. (This is also the expected revenue that the seller can obtain if she uses a standard first- or second-price auction with no reserve price.) We use the “must sell” auction as a benchmark for evaluating the revenue gains from the optimal allocation rule. These gains are substantial. In an example with three buyers whose valuations are distributed uniformly on the unit interval, we find that the expected revenues to the first seller increases by 53% (relative to the third order statistic) when she uses the optimal mechanism. The second seller also benefits since the sale price of her good increases from $x(3)$ to $x(2)$ if the first seller does not allocate the good. Her expected revenue increases by 16%.

We also compare the performance of the optimal auction to a standard auction with an optimal reserve price. We first prove that the presence of the allocation externality implies that the standard first or second-price auction with a reserve price does not have a strictly increasing symmetric equilibrium. However, as Jehiel and Moldovanu [13] have shown, there is an equilibrium with partial pooling at the reserve price. We derive the partial-pooling equilibrium for our uniform example and compute the expected revenues from using an optimal reserve price. We find that a reserve price is not a very effective way for the seller to raise revenues. The gain in expected revenue is only 21% (compared to no reserve price), roughly half of the gain from the optimal auction. One reason is that the partial pooling
is a source of allocative inefficiency, since it implies that a buyer whose value is below the two highest may get the good. The other, more important reason is that the outside option of winning the second auction at a price below the reserve price causes the participation threshold in the first auction to be substantially higher than the optimal reserve price. In our example, only 40% of the bidders bid in the first auction and roughly half of them bid the reserve price. Clearly, the threat to withhold the good if the second-highest bid is too low relative to the third-highest bid is more effective than the threat to withhold if the highest bid is too low.

Our paper is the first to study optimal mechanism design in sequential auctions with competing sellers. The literature typically assumes that sellers are nonstrategic – they use a standard auction with a zero reserve price – and focuses on characterizing equilibrium bidding behavior. Milgrom and Weber [20] show that, in an IPV environment with \( N \) buyers who have unit demands, prices for \( k \) identical objects sold sequentially in first- or second-price forms a martingale and are on average equal to the expected value of the \((k + 1)\)-th order statistic.\(^3\) Black and de Meza [6] examine the impact of multi-unit demands on prices in sequential, second-price auctions in a model with two sellers and two identical goods. Budish and Zeithammer [7] use this setting to extend the Milgrom and Weber analysis to imperfect substitutes (and two-dimensional types). Kirkegaard and Overgaard [16] is the exception. They show that the early seller in the Black and de Meza model can increase her expected revenue by offering an optimal buy-out price. Our analysis allows the early seller to consider any mechanism, in the special case of unit demands.\(^4\)

This paper is related to the work on auctions with externalities, where the payoff to a losing bidder depends on whether and to whom the object is allocated, and to the work on type-dependent outside options, where bidders have private information about their payoff if they lose.\(^5\) Jehiel and Moldovanu [13] study the impact of interactions by buyers

\(^3\)There is a large empirical literature that tests the martingale prediction (e.g., Ashenfelter [1], Ashenfelter and Genesove [2], and Beggs and Graddy [5]). Ashenfelter and Graddy [3] provide a survey of this literature.

\(^4\)There is a growing literature (e.g., Backus and Lewis [4], Said [26], and Zeithammer [28]) that studies bidding behavior in sequential, second-price auctions in stationary environments where new buyers and sellers enter the market each period. These papers make behavioral assumptions that effectively rule out the allocation externality.

\(^5\)This paper is also related to the recent literature on optimal design of auctions (and disclosure rules) in which the externalities are due to resale (e.g., Calzolari and Pavan [9], Carroll and Segal [10], and Dworczak [11]).
in a post-auction market on bidding behavior in standard auctions. Figueroa and Skreta [12] and Jehiel et al. [14, 15] consider revenue-maximizing mechanisms in a more general model of externalities. In our setting, the payoff to a buyer who fails to get the first object depends both on his own type and on the highest value among the other losing buyers. A feature of our environment is that the optimal threat by the seller -- that is, the action that minimizes the continuation payoff of all non-participating buyers -- is to not allocate the object. A consequence is that the participation constraint binds only for the lowest type of buyer. The optimal threat in Figueroa and Skreta [12] and Jehiel et al. [14, 15] is more complicated, and calculating the “critical type” for whom the participation constraint binds can be challenging.

Finally, our paper contributes to the literature on competing mechanisms. Burguet and Sakovics [8] study the case of two sellers with identical goods who simultaneously choose reserve prices in second-price auctions. They find that competition for buyers lowers equilibrium reserve prices, but not to zero. McAfee [19] and Peters and Severinov [24] consider the general mechanism choice problem and show that, when the number of sellers and buyers in a homogeneous good market is large, second-price auctions with zero reserve prices emerge as an equilibrium mechanism. These results lead Peters [23] to conclude that competition among sellers promotes simple, more efficient mechanisms. Our results suggest that this conclusion may not apply when auctions are sequenced. The early seller in our model does not have to compete for buyers. When he uses the optimal withholding rule, all buyers participate because, in doing so, they increase the likelihood that the good is allocated and their chances of winning the subsequent auction. This is not the case when the seller tries to withhold the good using a simple reserve price. We discuss competition between sellers further in Section 6.

The organization of the rest of the paper is as follows. In Section 2 we present the model. In Section 3 we derive the optimal allocation rule. In Section 4 we derive the transfers and show they can be implemented using a modified third-price auction or a pay-your-bid auction with a rebate. In Section 5 we evaluate the gains from using the optimal mechanism by comparing it to a standard auction with and without an optimal reserve price. In Section 6, we consider extensions. Section 7 provides concluding remarks.
2 Model

There are $N$ ex ante identical potential buyers, indexed by $i$, with unit demand for an indivisible good. Each buyer $i$’s privately observed valuation for the good $X_i$ is independently drawn from distribution $F$ with support $[\underline{x}, \bar{x}], \underline{x} \geq 0$. We will sometimes refer to a buyer’s valuation as his type. We assume that $F$ has a continuous density $f$ and that the virtual valuation

$$\psi(x) \equiv x - \frac{1 - F(x)}{f(x)}$$

is increasing in $x$. Order the valuations from highest to lowest $X(1), X(2), \ldots, X(N)$.

There are two sellers who sell identical units of the good. Each seller sells one unit. The sales are sequential and take place over two periods. In the second period, the second unit is sold in a second-price auction with no reserve price. That is, the second seller is not a strategic player, and so we typically will refer to the first seller as just “the seller.” That seller’s valuation of the good is normalized to zero. This structure is common knowledge.

We will consider the revenue-maximizing mechanism for the seller in the first period, given that any buyer who does not obtain the first object will participate in the auction for the second object.

The restriction to two units simplifies the notation by letting us focus on the first three order statistics. Let

$$F_1(x) \equiv [F(x)]^N,$$

$$F_2(x) \equiv F_1(x) + N [F(x)]^{N-1} [1 - F(x)],$$

and

$$F_3(x) \equiv F_2(x) + \frac{N(N-1)}{2} [F(x)]^{N-2} [1 - F(x)]^2$$

denote the distributions of the first, second, and third order statistics $X(1), X(2),$ and $X(3)$, respectively. The corresponding densities are

$$f_1(x) \equiv N [F(x)]^{N-1} f(x),$$

$$f_2(x) \equiv N(N-1) [F(x)]^{N-2} [1 - F(x)] f(x),$$

and

$$f_3(x) \equiv \frac{N(N-1)(N-2)}{2} [F(x)]^{N-3} [1 - F(x)]^2 f(x).$$
It will also be useful to define the order statistics of the competing valuations that a single buyer faces. Order the valuations of the other $N-1$ buyers from highest to lowest $Y(1), Y(2), \ldots, Y(N-1)$. We denote the distributions of $Y(1)$ and $Y(2)$, respectively, by

$$G_1(x) \equiv [F(x)]^{N-1}$$

and

$$G_2(x) \equiv G_1(x) + (N-1) [F(x)]^{N-2} [1 - F(x)].$$

The corresponding densities are

$$g_1(x) \equiv (N-1) [F(x)]^{N-2} f(x)$$

and

$$g_2(x) \equiv (N-1)(N-2) [F(x)]^{N-3} [1 - F(x)] f(x).$$

Finally, let $F_2|_{x(1)}$ denote the distribution of the second order statistic $X(2)$ conditional on the value of the first order statistic $X(1)$ being $x(1)$:

$$F_2|_{x(1)}(x) \equiv [F(x)]^{N-1} / \left[ F(x(1)) \right]^{N-1},$$

with density

$$f_2|_{x(1)}(x) \equiv (N-1) [F(x)]^{N-2} f(x) / \left[ F(x(1)) \right]^{N-1}.$$ 

Define the distributions $F_3|_{x(2)}$ and $G_2|_{y(1)}$ and their densities analogously.

3 The Optimal Mechanism

In our model, it is a weakly dominant strategy for any buyer who did not obtain the first object to submit a bid equal to his valuation in the second period auction. Thus, in designing his mechanism, the seller does not have to be concerned about the leakage problem. Any information buyers acquire in the first period about the types of competitors does not influence their bidding behavior in the second period. As a result, buyers have no incentive to bid untruthfully in period one to affect behavior in period two. However, in period one, a buyer’s bid may still influence the allocation of the first object, which does affect outcomes in the second period. The design of the optimal mechanism for the seller must take that incentive into account.
The payoff to a buyer $i$ with valuation $X_i$ in the second period, provided that he did not obtain the first object, depends on whether or not the first object was allocated to the competitor with the highest type $Y(1)$. If so, then buyer $i$’s payoff, \( \max \{ X_i - Y(2), 0 \} \), is a function of the highest remaining competitor’s type $Y(2)$. If not, then buyer $i$’s payoff is \( \max \{ X_i - Y(1), 0 \} \). All else equal, buyer $i$ prefers that the first object go to his strongest competitor so that competition in the subsequent auction is reduced. Thus, the expected payoff to a buyer depends on the two highest valuations \( y \geq z \) among his competitors.

Let \( x \in [\underline{x}, \overline{x}]^N \) denote the vector of reported types, and let \( P_i(x) \) denote the probability that the first object is assigned to buyer $i$ given $x$. We denote the highest-type competitor of bidder $i$ by $j(1)$ (so that $X_{j(1)} = y$). Then, the expected payoff to a bidder $i$ with type $x_i$ given vector of reports $x$, excluding any payment to the first seller, is

\[
P_i(x) \cdot x_i + P_{j(1)}(x) \cdot \max \{ x_i - z, 0 \} + (1 - P_i(x) - P_{j(1)}(x)) \cdot \max \{ x_i - y, 0 \}.
\]

To interpret Expression 1, observe that if $x_i$ is not one of the two highest valuations (if $z > x_i$), then bidder $i$ gets a payoff only if he receives the first object. If bidder $i$ has the second-highest valuation (if $y > x_i > z$), then he again receives his valuation if the first object is allocated to him, but he also gets payoff $x_i - z$ from winning the second auction if the first object goes to bidder $j(1)$. Finally, if $x_i$ is the highest valuation (if $x_i > y$), then bidder $i$ either 1) gets the first object himself, 2) gets the second object at price $z$ if the first object goes to bidder $j(1)$, or 3) gets the second object at price $y$ in any other case.

The next steps involves using the first-order incentive compatibility constraints to express the transfer payments from buyers in terms of their payoffs and the allocation rule and then choose the allocation rule that maximizes the sum of the payments. The standard approach defines the payoffs and allocation rule in terms of the vector of reported types. However, in our case, the bidder’s payoff depends not only upon reported types but also upon the highest actual types among his competitors. This dependence creates problems summing Expression 1 across bidders because these types vary with the identity of the bidder. To deal with this issue, we exploit the symmetry of the bidders and re-define payoffs and allocations in terms of the vector of reported realizations of order statistics. For any vector of reported types $x$, define $\hat{x}$ as the vector of reported types ordered from highest to lowest (with ties broken arbitrarily). Thus, the $k^{th}$ element of $\hat{x}$ is the $k^{th}$ highest reported type in $x$ (i.e., $\hat{x}_k = x_{(k)}$). Let $g$ denote the joint density of $\hat{x}$. The joint distribution of $\hat{x}_{-k} = \{\hat{x}_1, \ldots, \hat{x}_{k-1}, \hat{x}_{k+1}, \ldots, \hat{x}_N\}$ conditional on the value of $\hat{x}_k$ is denoted by $g_{-k|x_k}$.
We begin with the allocation rule. For each \( k \in \{1, \ldots, N\} \), let \( \hat{p}^k(\hat{x}) \) denote the probability that the mechanism allocates the object to the bidder with the \( k \)-th highest report, given \( \hat{x} \). Assuming that buyers report truthfully, we derive from \( \hat{p} \) other probabilities to simplify the notation. Let \( \hat{x}_1 \geq \hat{x}_2 \geq \hat{x}_3 \) be the highest, second highest, and third highest reported types. Then, for any \((a, b, c) \in [x, \bar{x}]^3\) such that \( a \geq b \geq c \), define

\[
p_1(a, b, c) \equiv E [\hat{p}_1(\hat{x}) | \hat{x}_1 = a, \hat{x}_2 = b, \hat{x}_3 = c],
\]
\[
p_2(a, b, c) \equiv E [\hat{p}_2(\hat{x}) | \hat{x}_1 = a, \hat{x}_2 = b, \hat{x}_3 = c].
\]

That is, \( p_1(a, b, c) \) and \( p_2(a, b, c) \) are the probabilities that the item will be allocated to the buyer with the highest and second-highest reports, respectively, conditional on \( a \geq b \geq c \) being the three highest reports. The expectation is taken over the lower order statistics.

Next, let \( a \geq b \) be the two highest reports other than buyer \( i \)'s, and suppose that buyer \( i \) reports \( c < b \). Then the probability that the item will be allocated to buyer \( i \), conditional on those reports, is

\[
p_3(a, b, c) \equiv E [\hat{p}_k(i)(\hat{x}) | \hat{x}_1 = a, \hat{x}_2 = b, \hat{x}_k(i) = c],
\]

where \( k(i) \) is the rank of \( c \) in \( \hat{x} \). Note that the expectation in this case includes the rank of \( c \) (i.e., \( k(i) \in \{3, \ldots, N\} \) is a random variable).

Assuming that other buyers report truthfully, we can then write the expected payoff to a buyer of type \( x \) who reports truthfully as

\[
\Pi(x|x) = \frac{x}{\bar{x}} \int_{\bar{x}}^{x} \left[ x - y + p_1(x, y, z)y + p_2(x, y, z)(y - z) \right] g_{2|y}(z) g_1(y) + \int_{\bar{x}}^{\hat{x}} \left( \int_{\bar{x}}^{x} \left[ p_1(y, x, z)(x - z) + p_2(y, x, z)x \right] + \int_{\bar{x}}^{y} \left[ p_3(y, z, x) \right] \right) g_{2|y}(z) g_1(y),
\]

where \( y \) and \( z \) represent, respectively, the highest and second-highest reports from rivals. More generally, we show in the appendix how to derive the payoff \( \Pi(q|x) \) to a buyer of type \( x \) who falsely reports his type as \( q \). We further show that \( \Pi(q|x) \) is convex in the buyer’s valuation; that is, \( \Pi_{22}(q|x) \geq 0 \).

The next steps are standard. Let \( t(q) \) be the expected transfer to the seller from a buyer who reports type \( q \). Incentive compatibility requires that buyers report their valuations truthfully, so the equilibrium payoff to a buyer of type \( x \) is

\[
U(x) = \max_q \Pi(q|x) - t(q).
\]
As the maximum of convex functions, $U(x)$ also is convex. It is therefore absolutely continuous and so differentiable almost everywhere. By standard arguments, its derivative is given by $U'(x) = \Pi_2(x|x)$, and

$$U(x) = U(x) + \int_\bar{x}^x \Pi_2(x'|x')dx',$$

where $\Pi_2(x|x)$ is the partial derivative of $\Pi(q|x)$ with respect to the second argument (the buyer’s true type) evaluated at the truthful report. It is given by

$$\Pi_2(x|x) = G_1(x) + \int_\bar{x}^x \left[ \int_\bar{x}^y \left[ p_1(y, x, z) + p_2(y, x, z) \right] + \int_\bar{x}^x p_3(y, x, z) \right] g_2(y, g_1) dy.$$  

Substituting $U(x) = \Pi(x|x) - t(x)$ into Expression 3 then yields

$$t(x) = t(x) + \Pi(x|x) - \Pi(x|x) - \int_\bar{x}^x \Pi_2(x'|x')dx'.$$

The mechanism is incentive compatible if $\Pi_2(q|x)$ is increasing in its first argument (the reported type). Because allocating the first object to any buyer weakly increases the total payoff to every buyer (ignoring any period-one transfer), withholding the first object minimizes the buyers’ payoffs. Thus, the period-one individual rationality condition is that $U(x)$ exceeds the expected payoff that a buyer of type $x$ could get from the second auction given that the first seller allocates his unit to no one. As usual, incentive compatibility implies that the mechanism is individually rational for all types if it is individually rational for a buyer of the lowest type $\bar{x}$: $t(x) \leq \Pi(\bar{x}|\bar{x})$.

The seller’s expected revenue is $N \cdot Et(X)$ where the ex ante expected transfer from a buyer is

$$Et(X) = t(x) - \Pi(x|x) + \int_\bar{x}^x \Pi(x|x)f(x) dx - \int_\bar{x}^x \int_\bar{x}^x \Pi_2(x'|x')dx'f(x) dx$$

$$= t(x) - \Pi(x|x) + \int_\bar{x}^x \left[ \Pi(x|x) - \frac{(1 - F(x))}{f(x)} \Pi_2(x|x) \right] f(x) dx.$$  

The first equality comes from using Expression 5 and the second from changing the order of integration of the double integral term. The bracketed term in the integral term represents the familiar virtual surplus that the seller can extract from each bidder. Substituting Expression 2 for $\Pi(x|x)$ and Expression 4 for $\Pi_2(x|x)$, the expected transfer can be expressed
in terms of virtual valuations as follows:

\begin{equation}
Et(X) = t(x) - \Pi(x|x) + \int_{x} f(x) \left\{ \int_{x} \left[ \psi(x) - y + p^1(x, y, z) + p^2(x, y, z)(y - z) \right] g_{2|y}(z) g_{1}(y) + \int_{x} \left[ p^1(x, y, z)(\psi(x) - z) + p^2(y, x, z)\psi(x) + p^3(y, z, x)\psi(x) \right] g_{2|y}(z) g_{1}(y) \right\} f(x)
\tag{7}
\end{equation}

In order to sum across bidders, we substitute the definitions of \( p^1, p^2, \) and \( p^3 \) into Expression 7. Note that the probability a given bidder has the \( k \)-th highest value is \( 1/N \) for each \( k \in \{1, \ldots, N\} \). Therefore, the expected transfer from each bidder is given by

\begin{equation}
Et(\hat{X}) = t(x) - \Pi(x|x)
+ \frac{1}{N} \int_{x} \int_{[\hat{x}, \hat{x}]^{N-1}} \left( \psi(\hat{x}) - \hat{x} + \hat{p}^1(\hat{x}, \hat{x}_{-1})\hat{x}_{2} + \hat{p}^2(\hat{x}, \hat{x}_{-1})(\hat{x}_{2} - \hat{x}_{3}) \right) g_{-1|\hat{x}_{1}}(\hat{x}_{-1}) f_{1}(\hat{x}_{1})
\tag{8}
\end{equation}

The seller maximizes expected revenue \( ER(\hat{X}) = NEt(\hat{X}) \) subject to incentive compatibility and individual rationality. To find the optimal allocation rule, we ignore the constraints and maximize the integral pointwise. Given any vector of ordered types \( \hat{x} \), taking the derivative of the seller’s expected revenue with respect to \( \hat{p}^k(\hat{x}) \) yields

\[
\frac{\partial ER(\hat{X})}{\partial \hat{p}^1(\hat{x})} = \hat{x}_{2}g_{-1|x_{1}(\hat{x}_{-1})}(\hat{x}_{-1})f_{1}(x_{1}) + [\psi(\hat{x}_{2}) - \hat{x}_{3}]g_{-2}(\hat{x}_{-2})f_{2}(\hat{x}_{2})
= [\hat{x}_{2} + \psi(\hat{x}_{2}) - \hat{x}_{3}]g(\hat{x}),
\]

\[
\frac{\partial ER(\hat{X})}{\partial \hat{p}^2(\hat{x})} = [\hat{x}_{2} - \hat{x}_{3}]g_{-1|x_{1}(\hat{x}_{-1})}(\hat{x}_{-1})f_{1}(\hat{x}_{1}) + \psi(\hat{x}_{2})g_{-2}(\hat{x}_{-2})f_{2}(\hat{x}_{2})
= [\hat{x}_{2} + \psi(\hat{x}_{2}) - \hat{x}_{3}]g(\hat{x}),
\]

and for all \( k > 2 \),

\[
\frac{\partial ER(\hat{X})}{\partial \hat{p}^k(\hat{x})} = \psi(\hat{x}_{k})g_{-k|x_{k}(\hat{x}_{-k})}(\hat{x}_{-k})f_{k}(\hat{x}_{k})
= \psi(\hat{x}_{k})g(\hat{x}).
\]
There are two things to note about the derivatives. First, the marginal revenue from increasing the probability $p_1$ of allocating to the highest bidder is exactly the same as from increasing the probability $p_2$ of allocating to the second highest bidder. Both are $x(2) + \psi(x(2)) - x(3)$. Intuitively, allocating the unit to either bidder means that both will obtain a good since the other bidder gets the second good at the third-highest valuation, $x(3)$. Not allocating the good means that only the highest bidder will get a good (the second one), and he will pay the second-highest valuation, $x(2)$. The difference in surplus between the first case ($x(1) + x(2) - x(3)$) and the second case ($x(1) - x(2)$) is $2x(2) - x(3)$. Leaving some surplus for the buyers to incentivize truth-telling results in replacing one of the $x(2)$ terms with the corresponding virtual valuation $\psi(x(2))$, and so the seller’s marginal revenue is $x(2) + \psi(x(2)) - x(3)$.

The second thing to note is that the marginal benefit from increasing $p_1$ or $p_2$ exceeds the marginal benefit from increasing the probability $p_k$ of allocating to any lower-ranked bidder $k > 2$. Since $x(2) \geq x(3)$ and the virtual valuation $\psi(\cdot)$ is increasing, we have

$$x(2) + \psi(x(2)) - x(3) \geq \psi(x(k)),$$

with strict inequality if $x(2) > x(k)$. The solution to the seller’s maximization problem, then, is to allocate to one of the top two bidders as long as

$$x(2) + \psi(x(2)) - x(3) \geq 0,$$

and not to allocate otherwise. That is, the reserve rule is a function of the second- and third-highest valuations. The unit is allocated for certain if $\psi(x(3)) \geq 0$ since in that case the inequality holds. If $x(2) + \psi(x(2)) < 0$, then the unit is certain not to be allocated. Otherwise, it may or may not be allocated, depending on the realization of the third order statistic. To maximize revenue, then, the seller should set $\hat{p}_1 + \hat{p}_2 = 1$ whenever $x(2) + \psi(x(2)) - x(3) \geq 0$, and should set $\hat{p}_k = 0$ for all $k$ otherwise.

We need to check that the solution to the relaxed problem satisfies the constraints. The above argument implies that $t(x) = \Pi(x|x) = 0$, so individual rationality for a buyer with the lowest possible valuation is satisfied. To satisfy incentive compatibility, we need to show that $\Pi_2(q|x)$ is increasing in $q$. It turns out that there is an additional subtlety relative to the standard mechanism design environment. Intuitively, the condition that $\Pi_2(q|x)$ is increasing in $q$ means that a bidder with a higher valuation prefers a higher report than a bidder with a lower valuation because the benefit from increasing the report increases with
the true valuation. The condition corresponds to the requirement that the total probability a bidder wins a unit, either the first or second object, is increasing in his report. Allocating to the second-highest bidder (conditional on the good being allocated at all) satisfies that condition, but allocating to the top bidder may not.

The reason that assigning the object to the bidder with the highest valuation may violate incentive compatibility comes from the fact that the condition for allocating the good, Expression 9, is decreasing in the third-highest report $x_{(3)}$. Consider bidder $i$ with valuation $x_i$. Reporting $x' < x_i$ can raise the probability that the first good is allocated if $x'$ is the third-lowest report. If the unit is assigned to the second-highest bidder, then allocating it does not help bidder $i$ in the second auction – assigning it does nothing to reduce competition in the second auction, because the highest valuation among the remaining bidders is unchanged. On the other hand, allocating the first unit to the highest bidder would reduce competition in the second auction. Thus, assigning the unit to the highest bidder can create a situation where, when $x_i$ is the second-highest value, bidder $i$ would gain from misreporting: if $x_i + \psi(x_i) < x_{(3)}$ but $x_{(3)} + \psi(x_{(3)}) > x'$. Reporting truthfully means that bidder $i$ does not get a good (the first unit will not be allocated and the highest bidder will get the second), but by reporting $x'$ bidder $i$ gets the second good (after the first good is allocated to the highest bidder).

Thus, allocating to the second-highest bidder rather than the first when Expression 9 is satisfied ensures that the mechanism is incentive compatible. (Details are in the appendix.) Theorem 1 summarizes the optimal mechanism.

**Theorem 1** If the distribution of buyer values $F$ has increasing virtual values, then the following is an optimal (direct) mechanism for the first seller. (Ties are broken randomly.)

1. **Allocation rule:** The seller allocates the good to the bidder with the second-highest valuation if

   $$x_{(2)} + \psi(x_{(2)}) - x_{(3)} \geq 0$$

   and does not allocate otherwise.

2. **Revenue:** The expected revenue to the seller is

   $$E[X_{(2)} + \psi(X_{(2)}) - X_{(3)}|X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0] \Pr\{X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0\}.$$
The expected revenue expression is obtained by substituting the optimal allocation rule into Expression 8, integrating, and recognizing that \( E[\psi(X_1)] = E[X_2] \). We can also write it in integral form. Let \( x_L \) denote the solution to
\[
x + \psi(x) = 0,
\]
and let \( x_H > x_L \) denote the solution to
\[
\psi(x) = 0.
\]
The allocation rule specifies that the object is always allocated when \( x_2 \geq x_H \) and never allocated when \( x_2 < x_L \). Then expected revenue is
\[
\int_{x_L}^{x_H} \int_{x_L}^{x_2 + \psi(x_2)} [x_2 + \psi(x_2) - x_3] f_3(x_2) f_2(x_2) dx_3 dx_2 + \int_{x_H}^{x} \int_{x_L}^{x_2} [x_2 + \psi(x_2) - x_3] f_3(x_2) f_2(x_2) dx_3 dx_2. \tag{10}
\]

An interesting benchmark for evaluating the revenue gains from using the optimal reserve rule is the expected revenue that the seller can obtain when he must sell the unit with probability 1. It follows from the above analysis that \( \hat{p}^2 = 1 \) in the optimal “must sell” mechanism and that the expected revenue of this mechanism is equal to
\[
E [X_2 + \psi(X_2) - X_3]. \tag{11}
\]
The follow lemma, which follows from Loertscher and Marx’s [18] Lemma 2, allows us to express that revenue in terms of expected values of order statistics.

**Lemma 2** \( E[\psi(X_2)] = 2E[X_3] - E[X_2] \).

Applying this lemma to Expression 11 yields \( E[X_3] \). This result is not too surprising. In our setting, Milgrom and Weber [20] show that the expected revenue to the seller who uses a first-price or second-price auction with no reserve is \( E[X_3] \). The optimal “must sell” mechanism is revenue equivalent to a first or second-price auction with no reserve price.

### 3.1 Example: Three bidders, uniform valuations

To illustrate the working of the optimal mechanism, suppose that there are three buyers whose valuations are distributed uniformly between zero and one. That is, \( N = 3 \) and
In that case, virtual valuations are given by $\psi(x) = 2x - 1$. The reserve rule is to allocate when

$$3x(2) - 1 > x(3).$$

Then the good is always allocated when $x(2) \geq x_H = \frac{1}{2}$ and never allocated when $x(2) < x_L = \frac{1}{3}$. Figure 1 illustrates the combinations of values of $x(2)$ and $x(3)$ that lead to allocation.

What is the probability that the unit is allocated? To calculate that probability, we integrate over the shaded area in Figure 1:

$$1 - F_2\left(\frac{1}{2}\right) + \int_{1/3}^{1/2} F_3|x(2)| (3x(2) - 1) f_2(x(2)).$$

Recall that $F_2$ is the distribution of the second order statistic $X(2)$ and $F_3|x(2)|$ is the distribution of $X(3)$ conditional on $X(2) = x(2)$. In this example,

$$F_2(x) = 3x^2 - 2x^3,$$

with density

$$f_2(x) = 6x(1 - x),$$
and
\[ F_{3|x(2)}(x) = \frac{x}{x(2)}, \]
with density
\[ f_{3|x(2)}(x) = \frac{1}{x(2)}. \]
Substituting these definitions into Expression 12 and integrating, we find the probability of allocation is \( \frac{23}{36} \approx 0.64. \)

Similarly, substituting \( f_2 \) and \( f_{3|x(2)} \) into Expression 10, the expected revenue to the seller is given by
\[
\begin{align*}
\hat{1}/2 \hat{1}/3 & \hat{1}/2 \hat{1}/3 \\
= & \int_{1/3}^{1/2} \int_{3x(2)-1}^{x(2)} \left[ 3x(2) - 1 - x(3) \right] \frac{1}{x(2)} 6x(2)(1-x(2)) \\
& + \int_{1/2}^{1} \int_{0}^{x(2)} \left[ 3x(2) - 1 - x(3) \right] \frac{1}{x(2)} 6x(2)(1-x(2)).
\end{align*}
\]
Integrating these expressions yields expected revenue of \( \frac{55}{144} \approx 0.38. \) We can also compute the expected revenue to the second seller (see appendix) and find that it is \( \frac{125}{432} \approx 0.289. \) By contrast, in the absence of any reserve rule, both sellers earn \( E[X(3)] = 0.25. \) Thus, the second seller benefits when the first seller uses the optimal reserve rule, but most of the gain goes to the first seller.

### 4 Implementing the Optimal Mechanism

In this section, we derive the payments in the optimal mechanism and show that they can implemented using a modified third-price or pay-your-bid auction. The following notation will be useful. For \( x \in [\bar{x}, \bar{x}], \) define the value \( a(x) \in [\bar{x}, \bar{x}] \) as the solution to \( a + \psi(a) = x, \) and let \( \phi(x) \equiv \max \{ x, a(x) \}. \) Note that \( a(x) > x \) if and only if \( x < x_H. \) Using that notation, the reserve rule in Expression 9 specifies that the first object is allocated when the second-highest value \( x(2) \) is at least \( \phi(x(3)). \)

We use Expression 5 to calculate the expected transfer of a buyer with type \( x. \) First consider the case in which the object is allocated: \( x(2) \geq \phi(x(3)). \) Then the gross payoff to the bidder with the highest type is \( x(1) - x(3). \) The derivative of his payoff with respect to his type, \( \Pi_2(x'|x'), \) is 1 if the bidder receives either of the objects and 0 otherwise. That is, \( \Pi_2(x'|x') = 1 \) if \( x' \geq \phi(x(3)) \) and \( \Pi_2(x'|x') = 0 \) for \( x' < \phi(x(3)). \) Plugging those values into
Expression 5 yields the following payment for the bidder with the highest type:

\[ x_{(1)} - x_{(3)} - \int_{\phi(x_{(3)})}^{x_{(1)}} dx' = \phi(x_{(3)}) - x_{(3)}. \]

Similarly, the gross payoff \( \Pi(x_{(2)}|x_{(2)}) \) to a bidder with the second-highest type is \( x_{(2)} \), and again \( \Pi_2(x'|x') = 1 \) if \( x' \geq \phi(x_{(3)}) \) and \( \Pi_2(x'|x') = 0 \) for \( x' < \phi(x_{(3)}) \). The resulting payment for the bidder is

\[ x_{(2)} - \int_{\phi(x_{(3)})}^{x_{(2)}} dx' = \phi(x_{(3)}). \]

For all other bidders, the payment is 0.

The other case to consider is when the first object is not allocated, \( x_{(2)} < \phi(x_{(3)}) \). In that case, the gross payoff \( \Pi(x_{(1)}|x_{(1)}) \) to the bidder with the highest type is \( x_{(1)} - x_{(2)} \). Now \( \Pi_2(x'|x') \) is 1 if \( x' \geq x_{(2)} \) and \( \Pi_2(x'|x') = 0 \) otherwise. Thus, the payment from the highest type is

\[ x_{(1)} - x_{(2)} - \int_{x_{(2)}}^{x_{(1)}} dx' = 0. \]

The payments for the other bidders also are 0.

We summarize those payments below, where \( t_k(x) \) denotes the transfer from the \( k \)-th highest bidder given vector of reported types \( x \):

- If \( \psi(x_{(3)}) < 0 \) and the item is allocated \( x_{(2)} \geq a(x_{(3)}) \), then
  \[ t^1(x) = a(x_{(3)}) - x_{(3)} > 0, \quad t^2(x) = a(x_{(3)}) > 0, \quad \text{and} \quad t^k(x) = 0 \ \forall k \geq 3. \]  
  \[ (13) \]

- If \( \psi(x_{(3)}) \geq 0 \), then the item is allocated (since \( x_{(2)} \geq x_{(3)} \)), and transfers are
  \[ t^1(x) = 0, \quad t^2(x) = x_{(3)} > 0, \quad \text{and} \quad t^k(x) = 0 \ \forall k \geq 3. \]  
  \[ (14) \]

- Finally, if the item is not allocated \( x_{(2)} < \phi(x_{(3)}) \), then
  \[ t^k(x) = 0 \ \forall k. \]  
  \[ (15) \]
It is straightforward to implement those payments and the optimal reserve rule in a version of a third-price auction. Define the modified third-price auction as follows: each buyer submits a bid in \([x, \bar{x}]\). As a function of the vector of bids \(b\), the good is allocated to the second-highest bidder if and only if \(b(2) \geq a(b(3))\). If the unit is not allocated, then no one makes any payments. If the unit is allocated, then the payments are based on the third-highest bid, \(b(3)\). When \(\psi(b(3)) > 0\), the highest bidder pays nothing and the second-highest bidder pays \(b(3)\); when \(\psi(b(3)) < 0\), then the highest bidder pays \(a(b(3)) - b(3) > 0\) and the second-highest bidder pays \(a(b(3))\).

**Theorem 3** If the distribution of buyer values \(F\) has increasing virtual valuations, then truthful bidding is an equilibrium of the modified third-price auction, and that equilibrium yields the optimal expected revenue for the first seller.

In fact, truthful reporting is a weakly dominant strategy. Consider, for example, the highest-valuation buyer in the case where \(\psi(x(3)) < 0\) and the item is allocated \(x(2) \geq a(x(3))\). Truthfully bidding \(b = x(1)\) yields a payoff of

\[
x(1) - x(3) - [a(x(3)) - x(3)] = x(1) - a(x(3));
\]

the bidder transfers \(a(x(3)) - x(3)\) to the first seller and then wins the second auction at price \(x(3)\). Any bid above \(x(2)\) yields that same payoff. A bid between \(a(x(3))\) and \(x(2)\) also results in payoff \(x(1) - a(x(3))\): the bidder gets the first item and transfers \(a(x(3))\) to the first seller. Any bid below \(a(x(3))\) gives a lower payoff, \(x(1) - x(2)\), since the first item will not be allocated, no transfers will be made to the first seller, and the bidder will win the second item at price \(x(2)\). The other cases are similar.

### 4.1 Pay-Your-Bid Auction

It is also possible to implement the optimal mechanism using a modified first-price or “pay your bid” auction, although the construction is more complicated. Expressions 13-15 show that a bidder’s payment depends on whether his is the highest or second-highest bid, but he submits only a single bid. One solution is to implement the highest bidder’s transfer as an unconditional (i.e., regardless of whether or not the good is allocated) payment together with a rebate equal to the winning price in the second auction.

More formally, we can construct a pay-your-bid auction with the following allocation and transfer rules. A buyer with valuation \(x\) submits a bid of \(\beta(x)\). The seller allocates the
object according to Expression 9, the rule from the optimal mechanism. It will turn out that \( \beta(\cdot) \) is strictly increasing, so the seller can implement that rule. If the item is allocated, then both the highest and second-highest bidders pay their bids. If the item is not allocated, then only the highest bidder pays his bid. In either case, the highest bidder then gets a rebate equal to the sale price of the second item \( x(3) \) if the first item is allocated, \( x(2) \) if it is not) assuming that he wins the second auction. The highest bidder does not get a rebate if he does not win the second auction.

The first step in our construction of the equilibrium bid function consists of specifying the probability that a bidder pays his bid. Let \( H(q) \) denote the probability that a buyer who submits a bid of \( \beta(q) \) in the auction will pay his bid, conditional on other buyers bidding according to \( \beta(\cdot) \). The bidder pays his bid for sure if \( \beta(q) \) is the highest bid \( (q > Y(1)) \); the probability of this event is \( G_1(q) \). (Recall that \( G_k \) is the distribution of \( Y(k) \), the \( k \)-th highest valuation among the \( N-1 \) competitors facing a single bidder.) The other event in which he pays his bid is when \( \beta(q) \) is the second-highest bid \( (Y(1) > q > Y(2)) \) and the good is allocated. The probability of this event for \( q > x_H \) is

\[
\int_q^{x_H} G_{2|y(1)}(q) g_1(y(1)) = (N-1) [F(q)]^{N-2} [1 - F(q)],
\]

and for \( x_H > q > x_L \) it is

\[
\int_q^{x_L} G_{2|y(1)}(q + \psi(q)) g_1(y(1)) = (N-1) [F(q + \psi(q))]^{N-2} [1 - F(q)].
\]

The good is not allocated when \( q < x_L \). Summing the probabilities of the two events in each of the three regions, we obtain

\[
H(q) = \begin{cases} 
G_2(q) & \text{if } q \geq x_H \\
G_1(q) + (N-1) [F(q + \psi(q))]^{N-2} [1 - F(q)] & \text{if } x_H > q \geq x_L \\
G_1(q) & \text{if } q < x_L
\end{cases}
\]

Let \( V(q, x) \) denote the expected payoff of a bidder of type \( x \) who bids \( \beta(q) \). When the bidder submits a truthful bid of \( \beta(x) \), then the probability that he gets a good is equal to the probability that he pays his bid, and so his expected payoff is

\[
V(x, x) = [x - \beta(x)] H(x).
\]
If $x$ is the highest type, then he pays his bid to the first seller and gets a refund from that seller for the price that he has to pay to acquire the good from seller 2. The price depends upon whether the good is allocated, but his payoff does not. If $x$ is the second-highest type, then he pays his bid only if the good is allocated to him and gets zero otherwise.

If the bidder deviates and submits a bid different from $\beta(x)$, then the probability of getting the good is no longer equal to $H(q)$ and the bidder’s expected payoff includes additional terms. These terms arise because the outcome in the second auction is based on the values of the bidders. Suppose that the bidder overstates his value by bidding more than he should in the first auction, i.e., $\beta(q) > \beta(x)$. If $\beta(q)$ is the highest bid, then he pays his bid to the first seller but he may lose the second auction and not get the rebate. This event will occur (conditional on $q > y_1(1)$) if (i) the good is allocated and $y_2 > x$ or (ii) the good is not allocated and $y_1 > x$. In these cases, the probability of getting a good is less than $H(q)$. Similarly, suppose that the bidder understates his value by bidding less than he should in the first auction, i.e., $\beta(q) < \beta(x)$. In this case, if his bid is the highest, then he always wins the second auction. But he may win the second auction even if his bid $\beta(q)$ is not the highest or second-highest bid. This event will occur (conditional on $q < y_2(2)$) if $x > y_1(1)$. We show in the appendix that these deviations yield lower payoffs than bidding truthfully.

Those additional terms also drop out when we evaluate the derivative of $V(q, x)$ with respect to $q$ at $q = x$. Taking that derivative yields the first-order condition

$$H'(x)[x - \beta(x)] - H(x)\beta'(x) = 0.$$  \hspace{1cm} (16)

As in the standard mechanism design environment, we can find the equilibrium bid function $\beta(\cdot)$ by solving the differential equation in Expression 16, with boundary condition $\beta(0) = 0$. To solve, rewrite Expression 16 as

$$\frac{d}{dx}[H(x)\beta(x)] = xH'(x).$$

Integrating over an interval $[a, b]$ yields

$$H(b)\beta(b) - H(a)\beta(a) = \int_a^b xH'(x).$$  \hspace{1cm} (17)

We then construct the solution piecewise by plugging in the values of $H(\cdot)$ and $H'(\cdot)$. For $x \in [\underline{x}, x_L]$,

$$\beta(x) = \frac{1}{G_1(x)} \int_\underline{x}^x sg_1(s).$$  \hspace{1cm} (18)
For $x \in (x_L, x_H]$, 
\[
\beta(x) = \frac{1}{H(x)} \left[ G_1(x_L)\beta(x_L) + \int_{x_L}^{x} sH'(s) \right]
\]
(19)

where 
\[
H'(s) = g_1(s) + (N-1)((N-2)(F(q+\psi(q))^{N-3}(1-F(q)(1+\psi'(s))f(s+\psi(s))-(F(q+\psi(q))^{N-2}f(s)).
\]

Finally, for $x \in (x_H, \bar{x}]$, 
\[
\beta(x) = \frac{1}{G_2(x)} \left[ G_2(x_H)\beta(x_H) + \int_{x_H}^{x} sg_2(s) \right].
\]
(20)

Because the bid function is invertible (it is strictly increasing), the seller can implement the optimal reserve rule. Note that even a buyer who has a valuation $x < x_L$, and who therefore knows that the mechanism will never assign him the first object, is willing to participate. If his is the highest valuation, then he will obtain the second good at a total cost equal to his bid $\beta(x)$ in the first auction. (Recall that he pays $\beta(x)$ and then is refunded the sale price $y(1)$ in the second auction.) Since $x < x_L$, his bid is given by 
\[
\beta(x) = \frac{1}{G_1(x)} \int_{x}^{x_L} y_1 g_1(y_1) = E[Y(1)|Y(1) \leq x],
\]
the expected sale price that he faces in the second auction, conditional on having the highest valuation. Thus, he is willing to bid in the first auction rather than wait for the second.

To demonstrate that this pay-your-bid auction implements the optimal mechanism, it remains only to show that it yields the optimal expected revenue; that is, we need to establish that 
\[
E \left[ \beta(X_{(1)}) - X_{(2)} \right] + 
E \left[ \beta(X_{(2)}) + X_{(2)} - X_{(3)} \right] \cdot X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0 \right] \cdot \Pr \left( X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0 \right)
= 
E \left[ \psi(X_{(1)}) - X_{(2)} \right] + 
E \left[ X_{(2)} + \psi(X_{(2)}) - X_{(3)} \right] \cdot X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0 \right] \cdot \Pr \left( X_{(2)} + \psi(X_{(2)}) - X_{(3)} \geq 0 \right).
\]

The proof is in the appendix. It is based on and is similar to the revenue equivalence proof of Riley and Samuelson [25] developed for the single seller case. Thus, we have the following result:

22
Theorem 4 If the distribution of buyer values $F$ has increasing virtual valuations, then the bid function specified in Expressions 18-20 is an equilibrium of the modified pay-your-bid auction, and that equilibrium yields the optimal expected revenue for the first seller.

In the auction, the transfer from the second-highest bidder is always positive. However, the realized transfer from the highest bidder may be negative, in which case the first seller makes a payment to the bidder. For example, suppose that the highest valuation $x_{(1)}$ is below $x_L$ and that $x_{(2)}$ is close to $x_{(1)}$ (in particular, $x_{(2)} > E[X_{(2)}|X_{(1)} = x_{(1)}]$). Then the item is not allocated, and the highest bid, $\beta(x_{(1)}) = E[X_{(2)}|X_{(1)} = x_{(1)}]$, is less than the refund of the second auction’s sale price, $x_{(2)}$: $\beta(x_{(1)}) - x_{(2)} < 0$.

In summary, the presence of a subsequent, competing auction creates several differences in the auctions that implement the optimal mechanism. The first is that the optimal rule for not allocating the good cannot be implemented by a reserve price. The second is that, if the good is allocated, then it is allocated to the second-highest bidder, not the highest-bidder. And, third, the seller receives payments from both the highest and second-highest bidders.

4.2 Example: Three bidders, uniform valuations

To demonstrate how the modified pay-your-bid auction works and to confirm that it yields the optimal revenue, we return to the example from Section 3.1: $N = 3$ and $F = U[0, 1]$. Virtual valuations are given by $\psi(x) = 2x - 1$, and the seller allocates when

$$3x_{(2)} - 1 > x_{(3)},$$

so the good is always allocated when $x_{(2)} \geq x_H = \frac{1}{2}$ and never allocated when $x_{(2)} < x_L = \frac{1}{3}$.

In our example,

$$H(x) = \begin{cases} 
G_2(x) = 2x - x^2 & \text{if } x \geq \frac{1}{2} \\
G_1(x) + (N - 1)\ [F(x + \psi(x))]^{N-2} [1 - F(x)] = 8x - 5x^2 - 2 & \text{if } \frac{1}{2} > x \geq \frac{1}{3} \\
G_1(x) = x^2 & \text{if } x < \frac{1}{3}
\end{cases}$$

$$G_1(x) = F(x)^2 = x^2,$$

$$G_2(x) = F(x)^2 + (N - 1)F(x)^{N-2}(1 - F(x)) = 2x - x^2,$$

and

$$H(x) = G_1(q) + (N - 1)F(q + \psi(q))^{N-2}(1 - F(q)) = 8x - 5x^2 - 2.$$
Using Expression 18, we have that the equilibrium bid function for \( x \in [0, \frac{1}{3}] \) is
\[
\beta(x) = \frac{2x}{3}.
\]

Using Expression 19, we compute the equilibrium bid function for \( x \in \left(\frac{1}{3}, \frac{1}{2}\right] \) as follows. The initial condition is
\[
G_1 \left(\frac{1}{3}\right) \beta \left(\frac{1}{3}\right) = \frac{1}{9} \cdot \frac{2}{9} = \frac{2}{81}.
\]
Substituting and integrating, we get
\[
\beta(x) = \frac{1}{H(x)} \left(\frac{2}{81} + \int_{1/3}^{x} sH'(s)ds\right) = \frac{4x^2 - \frac{10}{3}x^3 - \frac{24}{81}}{8x - 5x^2 - 2}.
\]
Finally, for \( x \in \left(\frac{1}{2}, 1\right] \), the initial condition is
\[
G_2 \left(\frac{1}{2}\right) \beta \left(\frac{1}{2}\right) = \frac{31}{108}
\]
and so, using Expression 20,
\[
\beta(x) = \frac{1}{2x - x^2} \left(\frac{31}{108} + \int_{1/2}^{x} s(2 - 2s)ds\right) = \frac{x^2 - \frac{2}{3}x^3 + \frac{13}{108}}{2x - x^2}.
\]
To summarize, the equilibrium bid function is
\[
\beta(x) = \begin{cases} 
  \frac{x^2 - \frac{2}{3}x^3 + \frac{13}{108}}{2x - x^2} & \text{if } x \in \left(\frac{1}{2}, 1\right] \\
  \frac{4x^2 - \frac{10}{3}x^3 - \frac{24}{81}}{8x - 5x^2 - 2} & \text{if } x \in \left(\frac{1}{3}, \frac{1}{2}\right] \\
  \frac{2x}{3} & \text{if } x \in [0, \frac{1}{3}] .
\end{cases}
\]
Note that \( \beta(x) \) is strictly increasing and continuous (since \( \lim_{x \downarrow 1/3} \beta(x) = \frac{7}{9} \) and \( \lim_{x \downarrow 1/2} \beta(x) = \frac{31}{81} \)). Figure 2 illustrates the bid function. Computing the expected revenue from the auction (net of the rebate to the high bidder) confirms that it is \(\frac{55}{144}\), the expected revenue of the optimal mechanism.

5 Standard Auctions with Reserve Prices

In this section, we compare the expected revenue of the optimal mechanism to the expected revenue of a standard auction with an optimal reserve price. The question is: how well does
a standard auction with an optimal reserve price do relative to the optimal auction with its more complicated reserve rule?

We begin by establishing that any symmetric equilibrium in the first auction involves pooling.

**Proposition 5** For any \( r \in (0, E[Y(1)]) \), there is no strictly increasing, symmetric pure-strategy equilibrium of either a first-price auction or a second-price auction with reserve price \( r \) for the first good.

To see why, consider a second-price auction with reserve price \( r \) and suppose that there is a symmetric equilibrium. The first-order condition for the optimal bid above \( r \) gives

\[
\tilde{\beta}(x) = \int_{\tilde{x}}^{x} y_2 g_{2|x}(y_2) = E[Y_2|Y_1 = x],
\]

the expected price in the second auction conditional on losing the first auction to another bidder of type \( x \). Let \( x^* \geq r \) denote the lowest valuation such that a buyer submits a bid. A buyer with valuation \( x^* \) must be indifferent between submitting a bid of \( \beta(x^*) \) in the first auction and not submitting a bid. (If he strictly preferred to bid, then so would nearby
types, and \( x^* \) would not be the lowest type to submit a bid.) The expected total payoff from submitting \( \beta(x^*) \) is

\[
(x^* - r)G_1(x^*) + \int_{x^*}^x \left[ \int_x^{x^*} [x^* - y(2)]g_2(y(1)) \right] g_1(y(1));
\]

the expected payoff from not submitting a bid is

\[
\int_x^{x^*} [x - y(1)]g_1(y(1)) + \int_{x^*}^x \left[ \int_x^{x^*} [x^* - y(2)]g_2(y(1)) \right] g_1(y(1)).
\]

The difference is zero when

\[
(x^* - r)G_1(x^*) = \int_x^{x^*} [x - y(1)]g_1(y(1));
\]

that is, when

\[
r = \frac{1}{G_1(x^*)} \int_x^{x^*} y(1)g_1(y(1)) = E[Y(1)|Y(1) \leq x^*].
\]

But that value of \( r \) is strictly greater than the value of \( \hat{\beta}(x^*) \) from Expression 21. (The former is the expectation of the highest of \( N - 1 \) valuations, conditional on all being below \( x^* \), while the latter is the expectation of the highest of \( N - 2 \), again conditional on all being below \( x^* \).) Thus, these two necessary conditions for equilibrium are incompatible, and we conclude that no strictly increasing, symmetric equilibrium of the second-price auction with reserve price \( r \) exists.

The non-existence result is not surprising. In our model, allocating the good in the first auction generates a positive externality for the losing buyers. Jehiel and Moldovanu [13] were the first to observe that a pure-strategy symmetric separating equilibrium does not exist in a second-price auction with positive externalities. However, they show that a symmetric equilibrium with partial pooling at the reserve price can exist. In that equilibrium, an interval of types \([x^*, x^{**}]\) all bid \( r \), types above \( x^{**} \) bid according to \( \hat{\beta}(x) \) (which is strictly increasing), and types below \( x^* \) do not bid. We construct such an equilibrium for our example and then calculate the optimal reserve price and revenues.

5.1 Three bidders, uniform valuations

We will derive the partial-pooling equilibrium for the \( N = 3, F = U[0, 1] \) case and compute the optimal reserve price \( r^* \) and associated revenue. The cutoff values \( x^* \) and \( x^{**} \) are
characterized by two indifference conditions. A buyer of type \( x^{**} \) is indifferent between bidding \( r \) (and tying with other types in \( [x^*, x^{**}] \)) and bidding just above \( r \); a buyer of type \( x^* \) is indifferent between bidding \( r \) and not bidding. Intuitively, the type-\( x^{**} \) buyer trades off overpaying for the first item relative to the expected price in the second auction when there is only one rival with a type in \( [x^*, x^{**}] \) against underpaying when there are two such rivals. The type-\( x^* \) buyer overpays when there are 0 or 1 rival with type in \( [x^*, x^{**}] \), but may get an item even when both rivals have higher types.

Solving the two indifference conditions (details are in the appendix) gives \( x^* = (1 + 1/\sqrt{3})r \) and \( x^{**} = (1 + 2/\sqrt{3})r \). We can now calculate the optimal reserve price \( r^* \). In the example, Expression 21 becomes \( \hat{\beta}(x) = x/2 \). The seller’s expected revenue is thus

\[
R_1(r) = [F_1(x^{**}) - F_1(x^*)] r + \int_{x^{**}}^{x^*} \left[ \frac{f_2(x_{(1)} f_2(x_{(2)})}{f_1(x_{(1)})} \right] f_1(x_{(1)})
\]

\[
= \left[ (x^{**})^3 - (x^*)^3 \right] r + \int_{x^{**}}^{x^*} \left[ \frac{(x^{**})^2}{(x_{(1)})^2} r + \frac{x_{(1)}}{2} \right] \frac{2x_{(2)}}{(x_{(1)})^2} \left( 3(x_{(1)}) \right)^2
\]

\[
= \frac{1}{4} - (x^{**})^3 + \frac{3}{4}(x^{**})^4 + r \left[ 3(x^{**})^2 - 2(x^{**})^3 - (x^*)^3 \right],
\]

where the last line follows from a lot of tedious calculations. Plugging in \( x^* = (1 + 1/\sqrt{3})r \) and \( x^{**} = (1 + 2/\sqrt{3})r \) and performing more tedious calculation gives

\[
R_1(r) = \frac{1}{4} + r^3 \left[ 2 - \frac{14}{3\sqrt{3}} + \frac{8}{\sqrt{3}} \right] - r^4 \left[ \frac{47}{12} + \frac{32}{3\sqrt{3}} - \frac{4}{\sqrt{3}} \right].
\]

We can now determine the optimal reserve price. Differentiating with respect to \( r \) yields the first-order condition

\[
3r^2 \left[ 2 - \frac{14}{3\sqrt{3}} + \frac{8}{\sqrt{3}} \right] + 4r^3 \left[ \frac{47}{12} + \frac{32}{3\sqrt{3}} - \frac{4}{\sqrt{3}} \right] = 0
\]

Solving for the optimal reserve price \( r^* \) yields

\[
r^* = \frac{3 \left[ 2 - \frac{14}{3\sqrt{3}} + \frac{8}{\sqrt{3}} \right]}{4 \left[ \frac{47}{12} + \frac{32}{3\sqrt{3}} - \frac{4}{\sqrt{3}} \right]} \approx 0.379.
\]

The corresponding values of \( x^* \) and \( x^{**} \) are \( x^* = (1 + 1/\sqrt{3})r^* \approx 0.60 \) and \( x^{**} = (1 + 2/\sqrt{3})r^* \approx 0.82 \).
Substituting the optimal reserve into the revenue function yields the maximal revenue, which is

\[
R_1(r^*) = \frac{1}{4} + (r^*)^3 \left[ 2 - \frac{14}{3\sqrt{3}} + \frac{8}{3\sqrt{3}} \right] - (r^*)^4 \left[ \frac{47}{12} + \frac{32}{3\sqrt{3}} - \frac{4}{\sqrt{3}} \right]
\]

\[
= \frac{1}{4} + \frac{27}{256} \left( \frac{2 - \frac{14}{3\sqrt{3}} + \frac{8}{3\sqrt{3}}}{\frac{47}{12} + \frac{32}{3\sqrt{3}} - \frac{4}{\sqrt{3}}} \right)^4 \approx 0.303.
\]

We can also compute the expected revenue for the second seller when the first seller sets the optimal reserve price. She gets the third-highest valuation if the first seller allocates and the second-highest valuation otherwise:

\[
R_2(r^*) = \int_0^{x^*} \left[ \int_0^{x(1)} x(2) f_2[x(1)] f_1(x(1)) \right] + \int_{x^*}^1 \left[ \int_0^{x(1)} x(3) f_3[x(3)] f_1(x(1)) \right] f_1(x(1))
\]

\[
= \int_0^{x^*} \left[ \int_0^{x(1)} x(2) \frac{2x(2)}{(x(1))^2} \right] 3(x(1))^2 + \int_0^{x^*} \left[ \int_0^{x(1)} x(3) \left(1 - \frac{x(3)}{x(1)}\right) \frac{2}{x(1)} \right] 3(x(1))^2
\]

\[
= \frac{1}{4} + \frac{1}{4}(x^*)^4 \approx 0.282.
\]

A striking feature of the equilibrium in the second-price auction with a reserve price is that the threshold for bidding, \(x^*\), is significantly higher than the optimal reserve price, \(r^*\). The outside option of winning the second auction at a price below \(r\) causes types between \(r\) and \(x^*\) not to bid in the first auction. Their lack of participation gives the high types an incentive to participate because they are more likely to win the first auction at price equal to \(r\). As a result, only 40% of the buyers bid in the first auction and roughly half of them bid the reserve price.

Table 1 summarizes the revenue results for our uniform example.

<table>
<thead>
<tr>
<th>Table 1: Revenue Comparisons</th>
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</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>Optimal Auction</td>
</tr>
<tr>
<td>Must-Sell Auction</td>
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<tr>
<td>Optimal Second Price Auction</td>
</tr>
</tbody>
</table>

In comparison to the must-sell auction, the optimal auction increases the expected revenues of both sellers. The first seller earns a 54% increase in revenues, and the second seller gets a 16% increase in revenues. A standard auction with an optimal reserve price
gives the second seller essentially the same increase but gives the first seller only a 20% increase in revenues. We conclude that reserve prices in standard auctions are not a very effective way for a seller to increase revenues in a sequential auction setting.

6 Extensions

The design of the optimal mechanism can be straightforwardly extended to environments in which either or both sellers have multiple units. For example, suppose that the first seller has one unit to sell and the second seller has $M$ units that she sells simultaneously in a uniform-price auction with no reserve. Buyers have a weakly dominant strategy to bid their value in second period so there is no leakage problem. Then the optimal allocation rule is to allocate the unit to the $(M + 1)$-th highest type if and only if $\psi(x_{(M+1)}) + M(x_{(M+1)} - x_{(M+2)}) > 0$. The first term is the virtual valuation of the marginal buyer (the one who gets an object if the first seller allocates and not otherwise), and the second term is the total savings to the $M$ buyers from reducing the price in the second auction from $x_{(M+1)}$ to $x_{(M+2)}$.

However, our analysis does not extend easily to situations in which the second seller can respond to the first seller’s mechanism. One issue is that the optimal response of the second seller may not be a second-price auction with a reserve price. When the first unit is allocated to the second-highest type, then the second seller faces a distribution of types that may not have increasing virtual valuations, even if the original distribution does. The inherited distribution is “hollowed out,” in the sense that a middle value is the one that gets removed. Further, the first seller’s optimal allocation rule implies that the types of the bidders remaining to face the second seller are correlated.

Another issue is that the solution to the first seller’s relaxed problem when the second seller sets a positive reserve price may not be the optimal mechanism. This solution can violate global incentive compatibility constraints. To illustrate the difficulty, we return to our three-bidder, uniform example and set $r$, the reserve price in the second auction, equal to 0.2. Figure 3 depicts the allocation rule obtained from solving the relaxed problem. Then are two cases to consider. First, if $x_{(2)} > r$, then the allocation rule is not to allocate if $3x_{(2)} - 1 < \max\{r, x_{(3)}\}$ and to allocate otherwise. Thus, in Figure 3, the good is not allocated in the triangle region $\triangle ACE$ where $x_{(3)} > r$ and the rectangle region $\square CDEF$ where $x_{(3)} < r$. Second, if $r > x_{(2)}$, then the allocation rule is to allocate if $x_{(1)} > r$ and not allocate otherwise. Thus, the good is allocated in the triangle region $\triangle 0CD$ if $x_{(1)} > 0.2$
Figure 3: Allocation region when the second seller sets a positive reserve price.

The incentive compatibility problem arises in this case.

To see why, suppose that a bidder has a type $x$ that is just above $r = 0.2$, and he is considering deviating to a report $x'$ just below $r$. If $x$ is the highest type, then he will get an item either way – the first item if he reports $x$, the second item if he reports $x'$. (Ignore the small probability that another bidder’s type is between $r$ and $x$). If $x$ is the lowest type, then he will get nothing with either report. But, if the highest type among other bidders, $y_{(1)}$, is greater than $x$, and the second-highest, $y_{(2)}$, is below $x$, then he gets $0$ from reporting $x$ (the first item will not be allocated and he will lose the second auction), while he gets $x - r$ from reporting $x'$ (the first item will be allocated to the highest bidder, and he will win the second item for $r$).

Carroll and Segal [10] encounter a similar problem in deriving the revenue-maximizing mechanism of a seller in an environment where the highest buyer type learns the types of the other buyers after the auction and has full bargaining power in the resale market. They introduce Lagrange multipliers for the incentive constraints into the seller’s maximization problem and prove that the solution to this constrained problem is optimal. They also show how to construct the withholding set. We think that their approach could be useful
in extending our analysis when \( r \in (0, 0.5) \). Interestingly, the incentive compatibility problem does not arise when \( r \geq 0.5 \). In this case, the optimal allocation rule reduces to the standard rule: allocate the good if \( \psi(x_{(1)}) \geq 0 \) and not allocate otherwise. This rule can be implemented with a reserve price.

The spillover effect between sequential sellers that we have identified is conceptually distinct from the problem of information leakage, but the two issues may interact. In general, the best response for the second seller depends on what information about the bidders’ types is disclosed after the first period. Consider the extreme case (as in Carroll and Segal [10]) where all private information is exogenously revealed after the first mechanism is run. If the second seller has all the bargaining power, then she will make a take-it-or-leave-it offer to the highest-type remaining buyer at exactly his value. Since the buyers anticipate that they will get no surplus in the second period, the problem facing the first seller is equivalent to the standard mechanism design environment. On the other hand, if the buyers have all the bargaining power in the second period, then the remaining buyer with the highest value will get the second item at a price equal to the second-highest remaining value, and so our optimal mechanism emerges as the equilibrium choice for the first seller.

7 Concluding Remarks

In sequential auction environments, losers of one auction can try to buy again, typically from a different seller. In this paper we show that for a seller who faces such competition from a subsequent auction, using a standard first- or second-price auction with a reserve price is suboptimal. Instead, we characterize the optimal mechanism, which features payments from the top two bidders and a reserve rule that depends on the second- and third-highest valuations. We also present a third-price auction and a pay-your-bid auction with a rebate that can be used in practice to implement the optimal mechanism.

Our setting is a special case of a mechanism design environment with externalities: when a buyer is awarded the first object, then he will not compete in the second auction. His absence, if he has the highest or second-highest valuation, increases the continuation payoff (that is, the payoff from the second auction) for the buyer with the highest remaining valuation. Our analysis of how a seller can increase revenues by accounting for this externality in the design of his auction is a first step towards an equilibrium analysis of competition between sellers in a sequential auction setting. Such an analysis is beyond the scope of this
paper but is an interesting subject for future research.
Appendix

8 Proving Theorem 1

8.1 Payoff from false report

Here we derive the payoff to a buyer of type \(x\) who falsely reports his type as \(q\). If \(q > x\), then

\[
\Pi(q|x) = \int_x^q \int_y^q \left\{ x - y + p^1(q, y, z)y + p^2(q, y, z)(y - z) \right\} g_{2|y}(z) g_1(y)
\]

\[+ \int_x^q \int_y^q \left[ p^1(q, y, z)x + p^2(q, y, z)(x - z) \right] g_{2|y}(z) g_1(y)
\]

\[+ \int_x^q \left[ p^1(y, q, z)(x - z) + p^2(y, q, z)x \right] g_{2|y}(z) g_1(y)
\]

If \(q < x\), then

\[
\Pi(q|x) = \int_x^q \int_y^q \left\{ x - y + p^1(q, y, z)y + p^2(q, y, z)(y - z) \right\} g_{2|y}(z) g_1(y)
\]

\[+ \int_x^q \int_y^q \left[ x - y + p^1(y, q, z)y + p^2(y, q, z)(y - z) \right] g_{2|y}(z) g_1(y)
\]

\[+ \int_x^q \left[ x - y + p^1(y, q, z)(y - z) + p^3(y, q, z)y \right] g_{2|y}(z) g_1(y)
\]

\[+ \int_x^q \left[ p^1(y, q, z)(x - z) + p^2(y, q, z)x \right] g_{2|y}(z) g_1(y)
\]

8.2 Convexity

The second derivative of the payoff \(\Pi(q|x)\) with respect to the second argument (the buyer's true type), \(\Pi_{22}(q|x)\), is given by

\[
\Pi_{22}(q|x) = \begin{cases} 
\int_x^q \left[ 1 - p^1(q, x, z) - p^2(q, x, z) \right] g_{2|x}(z) g_1(x) & \text{if } q \geq x \\
\int_x^q \left[ 1 - p^1(x, q, z) - p^2(x, q, z) \right] g_{2|x}(z) g_1(x) & \text{if } q < x 
\end{cases}
\]
Since \(1 - p^1 - p^2 - p^3 \geq 0\), \(\Pi_{22}(q|x) \geq 0\). That is, \(\Pi(q|x)\) is convex in the buyer’s valuation, as desired.

### 8.3 Incentive compatibility

Truthful reporting is a best response if for all \(x, q \in [\underline{x}, \bar{x}]\),

\[
U(x) = \Pi(x|x) - t(x) \geq \Pi(q|x) - t(q)
\]

\[
= U(q) + \Pi(q|x) - \Pi(q|q)
\]

\[
= U(q) + \int_q^x \Pi_2(q|x')dx'.
\]

(22)

If \(\Pi_2(q|x)\) is increasing in its first argument (the reported type), then the mechanism is incentive compatible: by substituting Expression 3 into Expression 22, we can rewrite the incentive compatibility condition as

\[
\int_q^x \Pi_2(x'|x')dx' \geq \int_q^x \Pi_2(q|x')dx'.
\]

That condition holds if \(\Pi_{21}(q|x) \geq 0\). The cross-partial derivative \(\Pi_{21}(q|x)\), evaluated at \(q \geq x\), is

\[
\Pi_{21}(q|x) = \\
\int_q^x \left[ \int_{\underline{x}}^x \left[ p_1^1(q, y, z) + p_1^2(q, y, z) \right] g_2|y(z) \right] g_1(y) + \int_q^x \left[ \int_{\underline{x}}^x \left[ p_2^1(y, q, z) + p_2^2(y, q, z) \right] g_2|y(z) \right] g_1(y),
\]

\[
\Pi_{21}(q|x) = \\
\int_q^x \left[ \int_{\underline{x}}^x \left[ p_2^1(y, q, z) + p_2^2(y, q, z) \right] g_2|y(z) \right] g_1(y) + \int_q^x \left[ \int_{\underline{x}}^x \left[ p_3^1(y, z, q) + p_3^2(y, z, q) \right] g_2|y(z) \right] g_1(y).
\]

(23)

where \(p_k^j(\cdot, \cdot, \cdot)\), \(j, k \in \{1, 2, 3\}\), represents the partial derivative of the probability \(p^k\) with respect to its \(j\)-th argument. At \(q < x\),

\[
\Pi_{21}(q|x) = \\
\int_x^q \left[ \int_{\underline{x}}^x \left[ p_2^1(y, q, z) + p_2^2(y, q, z) \right] g_2|y(z) \right] g_1(y).
\]

\[
\Pi_{21}(q|x) = \\
\int_x^q \left[ \int_{\underline{x}}^x \left[ p_2^1(y, q, z) + p_2^2(y, q, z) \right] g_2|y(z) \right] g_1(y).
\]

(24)

The functions \(p^1\), \(p^2\), and \(p^3\) corresponding to this mechanism are given by \(p^1(x, y, z) = p^3(x, y, z) = 0\) and

\[
p^2(x, y, z) = \begin{cases} 
1 & \text{if } y + \psi(y) - z \geq 0 \\
0 & \text{otherwise.} 
\end{cases}
\]

(34)
Plugging those functions into Expressions 23 and 24, we get

\[ \Pi_{21}(q|x) = \int_q^\infty \left( \int_{-\infty}^q p_2^2(y, q, z) g_{2|y}(z) \right) g_1(y) \]

if \( q \geq x \), and

\[ \Pi_{21}(q|x) = \int_0^q \left( \int_{-\infty}^0 p_2^2(y, q, z) g_{2|y}(z) \right) g_1(y) \]

if \( q < x \). Since \( p_2^2(y, q, z) \) is positive (the probability of allocating to the second-highest bidder is weakly increasing in the second-highest report), \( \Pi_{21}(q|x) \) is also positive. We conclude that the mechanism is incentive compatible, and Theorem 1 follows.

9 Proving Theorem 4

9.1 Equilibrium of the modified first-price auction

First, we show that the bid function \( \beta \) specified in Expressions 18-20 is an equilibrium of the modified pay-your-bid auction. Suppose that buyer \( i \)'s valuation is \( x_i \) and that all other buyers are bidding according to \( \beta \). We want to show that submitting a bid of \( \beta(x_i) \) is a best response for buyer \( i \). In particular, we want to show that buyer \( i \) cannot do better by submitting \( \beta(q) \) for some \( q \in [x, \bar{x}] \). (Bidding above \( \beta(\bar{x}) \) or below \( \beta(x) \) is dominated by bidding \( \beta(\bar{x}) \) or 0, respectively.)

For brevity, we will consider only the case where \( x_i \geq x_H \) and \( q > x_i \). The other cases are similar. If buyer \( i \) submits a bid of \( \beta(x_i) \), then his expected total (across both periods) payoff is

\[
\int_x^{x_i} x_i g_1(y_1) + \int_{x_i}^{\bar{x}} G_{2|y_1}(x_i) x_i g_1(y_1) - H(x_i) \beta(x_i) \\
= H(x_i)x_i - H(x_i)\beta(x_i). \tag{25}
\]

If he has the highest valuation, then he will pay the first seller \( \beta(x_i) \), win the second auction at a price equal to the third-highest valuation \( x_{(3)} \), and get a refund of \( x_{(3)} \) from the first seller. If he has the second-highest valuation, then he will get the first object (since \( x_i > x_H \), so \( x_i + \psi(x_i) > x_{(3)} \)) and pay \( \beta(x_i) \).
Submitting a bid of $\beta(q) > \beta(x_i)$ instead yields

\[
\int_x^{x_i} x_i g_1(y(1)) + \int_{x_i}^q G_{2|y(1)}(x_i) x_i g_1(y(1)) + \int_q^x G_{2|y(1)}(q) x_i g_1(y(1)) - H(q) \beta(q)
\]

\[
\leq \int_x^{x_i} x_i g_1(y(1)) + \int_{x_i}^q G_{2|y(1)}(q) x_i g_1(y(1)) - H(q) \beta(q)
\]

\[
= H(q)x_i - H(q)\beta(q). \tag{26}
\]

If he has the highest valuation, again he will get the second object for a total payment equal to his bid. If the highest valuation among his competitors, $y(1)$, is above $x_i$ but below $q$, then the object will be allocated (since $y(1) > x_i > x_H$) and buyer $i$ will pay his bid $\beta(q)$, but he will win the second auction only if the second-highest competitor’s valuation, $y(2)$, is less than his true valuation $x_i$. If $q$ is the second-highest bid, then buyer $i$ will get the first object and pay $\beta(q)$.

Subtracting Expression 26 from Expression 25, we get that the difference in payoff between bidding $\beta(x_i)$ and bidding $\beta(q)$ is greater than or equal to

\[
H(x_i)x_i - H(q)x_i + H(q)\beta(q) - H(x_i)\beta(x_i)
\]

\[
= H(x_i)x_i - H(q)x_i + \int_{x_i}^q xH'(x)
\]

\[
= H(x_i)x_i - H(q)x_i + H(q)q - H(x_i)x_i - \int_{x_i}^q H(x)
\]

\[
= H(q)(q - x_i) - \int_{x_i}^q H(x)
\]

\[
\geq 0,
\]

where the first equality uses Expression 17 and the second uses integration by parts. Thus, buyer $i$ cannot do better by bidding above $\beta(x_i)$. Similar arguments show that he cannot do better by bidding below $\beta(x_i)$ and that $\beta$ is a best response for buyers with valuations below $x_H$ as well.

### 9.2 Revenue equivalence

The proof that the modified first-price auction yields the optimal expected revenue is similar to the one that Riley and Samuelson [25] use to show that auctions in a broad class generate the same expected revenue in the standard mechanism design environment. As a preliminary, we use the definitions of $G_1$ and $G_2$ to rewrite the probability $H(q)$ that makes
a payment in the first auction as

\[
H(q) \equiv \begin{cases} 
[F(q)]^{N-1} + (N - 1)(1 - F(q))[F(q)]^{N-2} & \text{if } q \geq x_H \\
[F(q)]^{N-1} + (N - 1)(1 - F(q))[F(q + y(q))]^{N-2} & \text{if } x_H > q \geq x_L \\
[F(q)]^{N-1} & \text{if } q < x_L.
\end{cases}
\]

Let

\[
P(x) \equiv H(x)\beta(x)
\]

denote the expected payment of a bidder with a valuation of \(x\) (not counting the rebate to the high bidder). We want to show that

\[
N \cdot E[P(X)]
\]

\[
= E[\psi(X_1)] + E[\psi(X_2)X_2 + \psi(X_2) - X_3 \geq 0] \cdot Pr(X_2 + \psi(X_2) - X_3 \geq 0).
\]

From Expression 16, we know that equilibrium bids satisfy the first-order condition

\[
xH'(x) - P'(x) = 0
\]

for all \(x \in [x, \bar{x}]\). Individual rationality implies that \(P(0) = 0\), so we can integrate to get

\[
P(x) = \int_0^x x' H'(x')
\]

\[
= xH(x) - \int_0^x H(x'),
\]

where the second equality follows from integration by parts. Taking the expectation over \(x\) gives the \textit{ex ante} expected payment from a bidder to the first seller:

\[
E[P(X)] = \int_x^{\bar{x}} P(x)f(x) = \int_0^x P(x)f(x).
\]

Substituting for \(P(x)\) and integrating by parts, we obtain

\[
E[P(X)] = \int_0^x [xH(x) - \int_0^x H(x')dx'] f(x)dx
\]

\[
= \int_0^x [xf(x) + F(x) - 1] H(x)dx.
\]

That is, \(E[P(X)]\) equals

\[
= \int_0^xL [xf(x) + F(x) - 1] [F(x)]^{N-1}
\]

\[
+ \int_{x_L}^{x_H} [xf(x) + F(x) - 1] ((F(x))^{N-1} + (N - 1)(1 - F(x))[F(x + \psi(x))]^{N-2})
\]

\[
+ \int_{x_H}^{\bar{x}} [xf(x) + F(x) - 1] ([F(x)]^{N-1} + (N - 1)(1 - F(x))[F(x)]^{N-2})
\]

(27)
The first line of Expression 27 can be rewritten as
\[
\frac{1}{N} \int_0^{x_L} \left[ x - \frac{1-F(x)}{f(x)} \right] N[F(x)]^{N-1} f(x) = \frac{1}{N} \int_0^{x_L} \psi(x) f(x) f_1(x).
\]
(Recall that \( f_k \) is the density of the \( k \)-th order statistic.) Similarly, the third line of Expression 27 equals
\[
\frac{1}{N} \int_{x_H}^{x} \left[ x - \frac{1-F(x)}{f(x)} \right] N[F(x)]^{N-1} f(x) = \frac{1}{N} \int_{x_H}^{x} \psi(x) f_1(x) + \frac{1}{N} \int_{x_L}^{x} \psi(x) f_2(x).
\]
Finally, the second line of Expression 27 equals
\[
\frac{1}{N} \int_{x_L}^{x_H} \left[ x - \frac{1-F(x)}{f(x)} \right] N[F(x)]^{N-1} f(x) + N(N-1)(1-F(x))[F(x)]^{N-2} f(x)
\]
\[
= \frac{1}{N} \int_{x_L}^{x_H} \psi(x) f_1(x) + \frac{1}{N} \int_{x_L}^{x_H} \psi(x) f_2(x) \cdot \left( \frac{F(x+\psi(x))}{F(x)} \right)^{N-2}.
\]
Making those substitutions, we obtain that \( N \cdot E[P(X)] \) is equal to
\[
\int_0^{x_L} \psi(x) f_1(x) + \int_{x_L}^{x_H} \psi(x) f_2(x) \cdot \min \left\{ 1, \left( \frac{F(x+\psi(x))}{F(x)} \right)^{N-2} \right\}.
\]
Recall that the object is allocated when \( x(3) \leq x(2) + \psi(x(2)) \). As desired, then, the sum of expected payments is exactly the expectation of the highest virtual valuation \( \psi(X(1)) \), plus the expectation of the second-highest virtual valuation \( \psi(X(2)) \) weighted by the probability that the object will be assigned given the value of \( X(2) \).

10 Thresholds in Second-Price Auction with Reserve Price

To characterize the equilibrium, we introduce some notation. For \( k \in \{0,1,2\} \), define \( p_k(x^*, x^{**}) \) as the probability that a buyer has exactly \( k \) rivals with types between \( x^* \) and \( x^{**} \) and no rivals with a higher type:
\[
p_0 = (N-1)(N-2)[F(x^{**})-F(x^*)]^2 = (x^{**}-x^*)^2.
\]
\[
p_1 = (N-1)[F(x^{**})-F(x^*)][F(x^*)]^{N-2} = 2(x^{**}-x^*)x^*,
\]
\[
p_2 = \frac{(N-1)(N-2)}{2} [F(x^{**})-F(x^*)]^2 [F(x^*)]^{N-3} = (x^{**}-x^*)^2.
\]
Define $D_k$ as the expected value of the highest rival type in the second auction, conditional on $k$ and conditional on one of the $k$ rivals winning the first auction if $k > 0$:

$$D_0 = E[y_1 | y_1 < x^*] = \frac{2}{3} x^*,$$

$$D_1 = E[y_2 | y_1 \in [x^*, x^{**}], y_2 < x^*] = \frac{1}{2} x^*,$$

$$D_2 = \frac{1}{2} E[y_1 | y_1, y_2 \in [x^*, x^{**}]] + \frac{1}{2} E[y_2 | y_1, y_2 \in [x^*, x^{**}]] = \frac{1}{2} (x^{**} + x^*).$$

Finally, for $x \in [x^*, x^{**}]$, let

$$L(x) \equiv \int_{x^*}^{x^{**}} \int_{x}^{y(2)} [x - y(1)] g_{2[y(1), y(2)]} y(1) g_1(y(1))$$

denote the expected payoff in the second auction conditional on the winner of the first auction having a type above $x^{**}$, times the probability of that event. The dependence of $p_k$, $D_k$, and $L(x)$ on $x^*$ and $x^{**}$ is suppressed for readability.

The cutoff values $x^*$ and $x^{**}$ are characterized by two indifference conditions. A buyer of type $x^{**}$ is indifferent between bidding $r$ (and tying with other types in $[x^*, x^{**}]$) and bidding just above $r$; a buyer of type $x^*$ is indifferent between bidding $r$ and not bidding. That is,

$$p_0(x^{**} - r) + p_1 \left[ \frac{1}{2}(x^{**} - r) + \frac{1}{2}(x^{**} - D_1) \right] + p_2 \left[ \frac{1}{3}(x^{**} - r) + \frac{2}{3}(x^{**} - D_2) \right] + L(x^{**}) = p_0(x^{**} - r) + p_1(x^{**} - r) + p_2(x^{**} - r) + L(x^{**}; x^*, x^{**})$$

and

$$p_0(x^* - r) + p_1 \left[ \frac{1}{2}(x^* - r) + \frac{1}{2}(x^* - D_1) \right] + p_2 \left[ \frac{1}{3}(x^* - r) + L(x^*; x^*, x^{**}) \right] = p_0(x^* - D_0) + p_1(x^* - D_1) + p_2 \cdot 0 + L(x^*; x^*, x^{**}).$$

Taking differences, $x^{**}$ solves

$$p_1 \frac{1}{2}(D_1 - r) + p_2 \frac{2}{3}(D_2 - r) = 0$$

and $x^*$ solves

$$p_0(D_0 - r) + p_1 \frac{1}{2}(D_1 - r) + p_2 \frac{1}{3}(x^* - r) = 0.$$

The type-$x^{**}$ buyer trades off overpaying for the first item relative to the expected price in the second auction when there is only one rival with a type in $[x^*, x^{**}]$ against underpaying
when there are two such rivals. The type-$x^*$ buyer overpays when there are 0 or 1 rival with type in $[x^*, x^{**}]$, but may get an item even when both rivals have higher types.

Plugging in the values of $p_k$ and $D_k$ gives

$$x^* \left( \frac{1}{2} x^* - r \right) + (x^{**} - x^*) \frac{2}{3} \left( \frac{1}{2} (x^{**} + x^*) - r \right) = 0$$

(28)

and

$$(x^*)^2 \left( \frac{2}{3} x^* - r \right) + (x^{**} - x^*) x^* \left( \frac{1}{2} x^* - r \right) + \frac{1}{3} (x^{**} - x^*)^2 (x^* - r) = 0.$$  

(29)

The solutions to Expressions 28 and 29 are $x^* = (1 + 1/\sqrt{3})r$ and $x^{**} = (1 + 2/\sqrt{3})r$.

11 **Revenues to Seller 2 when Seller 1 uses Optimal Mechanism**

If seller 1 allocates his unit, then seller 2 earns $z$. Therefore,

$$\max_{1/3} \{ y, 3y - 1 \} \int_0^{1/3} \int_0^y \frac{z}{3y} dz dy (1 - y) \int_0^{1/3} \int_0^y \frac{z}{3y} dz dy (1 - y) dy$$

$$= 6 \int_{1/3}^{1/2} \frac{1}{2} (3y - 1)^2 (1 - y) dy + 6 \int_{1/3}^{1/2} \frac{1}{2} y^2 (1 - y) dy$$

$$= \frac{21}{108} \approx .194.$$

If seller 1 does not allocate his unit, then seller 2 earns $y$. Therefore,

$$\int_0^{1/3} \int_0^y \frac{1}{3} y f_3(z|y) f_2(y) + \int_{1/3}^{1/2} \int_0^y \frac{1}{3} y f_3(z|y) f_2(y)$$

$$= 6 \left( \int_0^{1/3} \int_0^y dz dy (1 - y) dy + \int_{1/3}^{1/2} \int_0^y dz dy (1 - y) dy \right)$$

$$= \frac{41}{432} \approx .095.$$

Therefore, the expected revenue for the second seller is

$$E \frac{21}{108} + \frac{41}{432} = \frac{125}{432} \approx 0.289.$$
References


